ANALYSIS OF THE LAMELLA ROOF

HORIZONTAL FORCES

by

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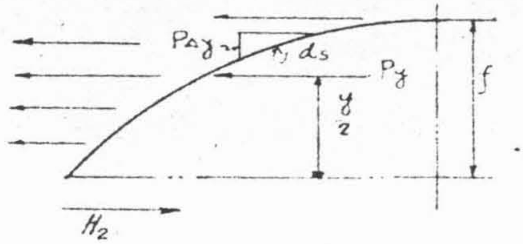
DETAILED CALCULATIONS
FOR
VERTICAL LOADING

$M_2 = -H_2 y + P\frac{y^2}{2}$ $\boxed{1A}$



For An Arch The Formula Is:

$$\int_{-a}^{a} \frac{M_m\, ds}{E.I} = \delta$$

Since $\delta = 0$ (Rib Shortening Neglected)

M = Unity = y And $ds \cong dx$, E And I Are Constant This Becomes

$$\int_{-a}^{a} M y\, dx = 0 \quad \boxed{1B}$$

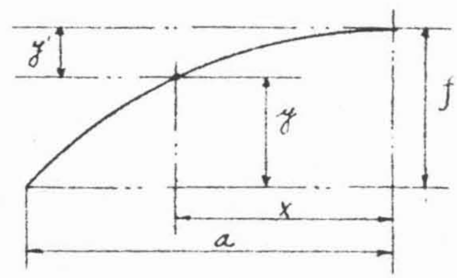
From The Parabola; $y = f(1-\frac{x^2}{a^2})$ Is Developed As Follows

Assume The Curve A Parabola With The Vertex At The Origin

$y' = kx^2$ ①

When $x = a$, $y' = f$ $\therefore f = ka^2$ ②

$k = \frac{f}{a^2}$



Combine ① And ②

$y' = f\frac{x^2}{a^2}$

But $y = f - y' = f - f\frac{x^2}{a^2} = f(1-\frac{x^2}{a^2})$

Substitute Value of M_2 From $\boxed{1A}$ in $\boxed{1B}$

$$= \int_{-a}^{a}\left[-H_2 y^2 + P\frac{y^3}{2}\right]dx = 0$$

Substitute $y = f(1-\frac{x^2}{a^2})$

$$\int_{-a}^{a}\left[-H_2 f^2\left(1-\frac{x^2}{a^2}\right)^2 + Pf^3\frac{\left(1-\frac{x^2}{a^2}\right)^3}{2}\right]dx = 0$$

Expanding $\displaystyle\int_{-a}^{a}\left[-H_2 f^2\left(1-2\frac{x^2}{a^2}+\frac{x^4}{a^4}\right) + P\frac{f^3}{2}\left(1-\frac{3x^2}{a^2}+3\frac{x^4}{a^4}-\frac{x^6}{a^2}\right)\right]dx = 0$

Intergrating $\displaystyle\left[-H_2 f^2\left(x-\frac{2x^3}{3a^2}+\frac{x^5}{5a^4}\right) + P\frac{f^3}{2}\left(x-\frac{3x^3}{3a^2}+\frac{3x^5}{5a^4}-\frac{x^7}{7a^6}\right)\right]_{-a}^{+a} = 0$

Clearing $\displaystyle 2\left[\frac{H_2 f^2 a(8)}{15} + Pf^3 a\,\frac{16}{70}\right] = 0$

Or $H_2 = \frac{3}{7}Pf$

Substituting In $\boxed{IA}$ $-M_2 = -H_2 y + \frac{Py^2}{2}$

$$M_2 = -\frac{3}{7} Pfy + \frac{1}{2} Py^2 \quad \boxed{1}$$

Substituting For y The Value $f = (1 - \frac{x^2}{a^2})$

$$M_2 = -\frac{3}{7} Pf^2 (1 - \frac{x^2}{a^2}) + \frac{1}{2} Pf^2 (1 - \frac{x^2}{a^2})^2$$

$$= -\frac{3}{7} Pf^2 + \frac{\frac{3}{7} Pf^2 x^2}{a^2} + \frac{1}{2} Pf^2 - Pf^2 \frac{x^2}{a^2} + \frac{1}{2} Pf^2 \frac{x^4}{a^4}$$

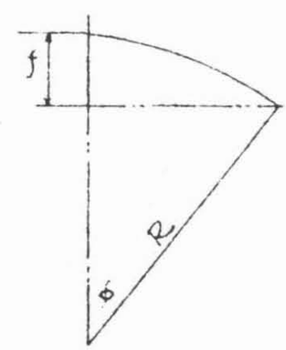
$$= Pf^2 \left[-\frac{3}{7} + \frac{3}{7} \frac{x^2}{a^2} + \frac{1}{2} - \frac{x^2}{a^2} + \frac{x^4}{2a^4} \right]$$

$$= \frac{Pf^2}{14} \left[-6 + 6 \frac{x^2}{a^2} + 7 - \frac{14x^2}{a^2} + 7 \frac{x^4}{a^4} \right]$$

$$\therefore M_2 = \frac{Pf^2}{14} \left[1 - \frac{8x^2}{a^2} + 7 \frac{x^4}{a^4} \right] \quad \boxed{2}$$

The Total Horizontal Reaction For The Live Load

$$H_l = H_1 + H_2 = PR \cos\phi + \frac{3}{7} Pf$$



$$\text{But} \quad \frac{R - f}{R} = \cos\phi$$

$$\therefore H_l = PR \frac{R-f}{R} + \frac{3}{7} Pf$$

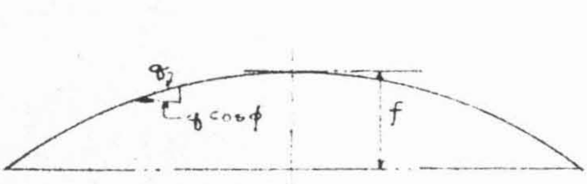
$$= \frac{PR^2 - PRf}{R} + \frac{3}{7} Pf = PR - Pf + \frac{3}{7} Pf$$

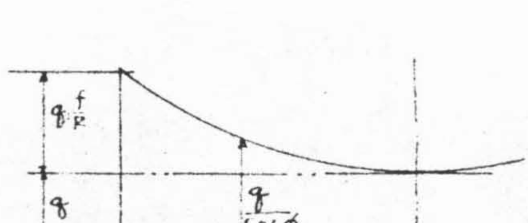
$$\therefore H_l = PR - \frac{4}{7} Pf \quad \boxed{3}$$

DEAD LOAD

Load per Unit Length of Horizontal Projection Is $\frac{q}{\cos\phi}$

For $\phi = \pm \phi_0$ (At The Hinges) $= \frac{q}{\cos\phi_0} = \frac{qR}{R-f} \cong q(1 + \frac{f}{R})$





THRUST - LIVE LOAD:-

The Thrust T_e At Any Arbitrary Point X is

$T_e = \sqrt{H^2 + V_x^2}$ $\qquad V_x = Px$

The Thrust At The Sills. Substitute In The Above Equation The value
$H_e = PR - \frac{4}{7}pf$ And for $X = a$

$T_o = p\sqrt{(R - \frac{4}{7}f)^2 + a^2}$

$T_o = p\sqrt{R^2 - 2R\frac{4}{7}f + \frac{16}{49}f^2 + a^2}$

Since $f = \frac{a^2}{2R}$; $a^2 = 2Rf$

Substituting

$T_o = p\sqrt{R^2 - \frac{8}{7}Rf + 2Rf + \frac{16}{49}f^2}$

$= p\sqrt{R^2 + \frac{6}{7}Rf + \frac{16}{49}f^2}$

$= p\sqrt{R^2 + \frac{6}{7}Rf + \frac{9}{49}f^2 + \frac{7}{49}f^2}$

$= p\sqrt{(R + \frac{3}{7}f)^2 + \frac{7}{49}f^2}$

$\frac{7}{49}f^2$ Is Neglible in Comparison With The First Term so May Be Neglected (Approx $\frac{1}{10}$ of 1%)

$\therefore T_o = P(R + \frac{3}{7}f)$

The Degree of Approximation Is Shown Thus:-

When $f = \frac{1}{10}R \quad \frac{qR}{R-f} = q\,1.111$ And $q\left(1+\frac{f}{R}\right) = q\,1.1 = 99\%$ Exact.

$f = \frac{1}{5}R \quad \frac{qR}{R-f} = q\,1.25$ And $q\left(1+\frac{f}{R}\right) = q\,1.2 = 96\%$ Exact.

$f = \frac{1}{6}R \quad \frac{qR}{R-f} = q\,1.20$ And $q\left(1+\frac{f}{R}\right) = q\,1.167 = 97.15\%$ Exact.

This Percentage of Error, When The Total Live And Dead Load Is Considered, For Roofs of $f = \frac{1}{6}R$ With a Thirty Pound Live Load Is Approximately One Half of One Percent Error.

The Dead Load Is Thus Assumed As A Load q Uniformly Distributed over The Horizontal Projection Plus An Additional Load. The Additional Load is $q \cdot \frac{f}{R} \cdot \frac{x^2}{a^2}$ The Maximum Value of $q \cdot \frac{f}{R}$ At The Hinges.

Considering The Half Arch Between $X=0$ And $X=a$ The Moment From The Additional Load Becomes:-

$$M_4 = M_o + H_4(f-y) - \int_0^x \left(q\frac{f}{R}\cdot\frac{x'^2}{a^2}\right)(x-x')\,dx'$$

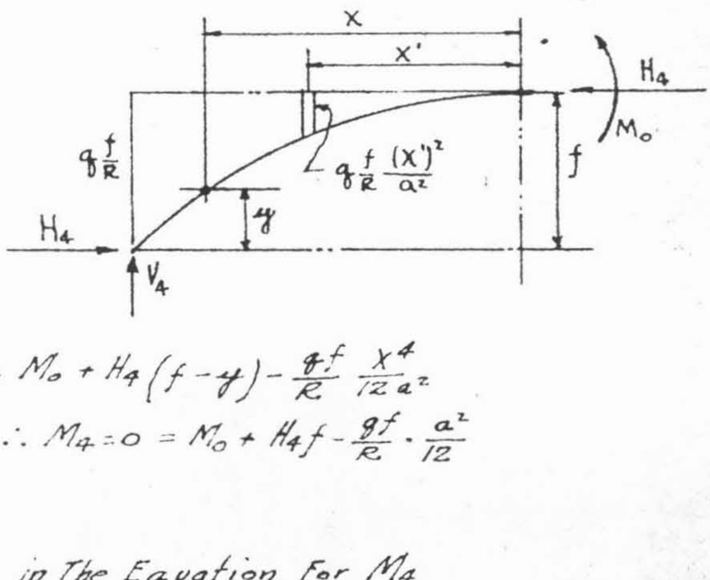
Intergrating The Second Half of The Equation

$$\int_0^x \left[q\frac{f}{R}\,\frac{(x')^2}{a^2}(x-x')\right]dx'$$ Where X is The Constant, X' is The Variable

$$= \int_0^x \left[q\frac{f}{Ra^2}\left(x(x')^2-(x')^3\right)\right]dx'$$

$$= \frac{qf}{Ra^2}\left[x\frac{(x')^3}{3}-\frac{(x')^4}{4}\right]_0^x$$

$$= \frac{qf}{Ra^2}\left[\frac{x^4}{3}-\frac{x^4}{4}\right] = \frac{qf\,x^4}{12Ra^2}$$



Hence ΣM To The Right of $A = M_o + H_4(f-y) - \frac{qf}{R}\,\frac{x^4}{12a^2}$

When $x = -a$: $y = 0$, $M_4 = 0 \quad \therefore M_4 = 0 = M_o + H_4 f - \frac{qf}{R}\cdot\frac{a^2}{12}$

And $M_o = -H_4 f - \frac{qf}{R}\cdot\frac{a^2}{12}$

Substitute This Value of M_o in The Equation For M_4

$$M_4 = -H_4 f + \frac{qf}{R}\,\frac{a^2}{12} + H_4 f - H_4 y - \frac{qf}{R}\,\frac{x^4}{12a^2}$$

$$= -H_4 y + \frac{qf}{R12}\left(a^2 - \frac{x^4}{a^4}\right)$$

$$\therefore M_4 = -H_4 y + \frac{qfa^2}{R12}\left(1-\frac{x^4}{a^4}\right) \qquad [4A]$$

Again $\int_{-a}^{a} My\,dx = 0$; $y = f\left(1-\frac{x^2}{a^2}\right)$; $\frac{ya^2}{f} = a^2 - x^2$; $x^2 = a^2\left(1-\frac{y}{f}\right)$

Solving For H_4 Using The Above Valve of M_4

$$M = -H_4 y + \frac{q f a^2}{12R}\left(1-\frac{x^4}{a^4}\right) = -H_4 y + \frac{q f a^2}{12R}\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{a^2}\right)$$

$$= -H_4 y + \frac{q a^2}{12R} y\left(1+\frac{x^2}{a^2}\right) = -H_4 y + \frac{q a^2}{12R} y\left[1+\left(1-\frac{y}{f}\right)\right]$$

$$= -H_4 y + \frac{q a^2}{6R} y - \frac{q a^2 y^2}{12R\,f}$$

$$\therefore \int_{-a}^{a} My\,dx = \int_{0}^{a}\left[-H_4 y^2\,dx + \frac{q a^2}{6R} y^2\,dx - \frac{q a^2}{12Rf} y^3\,dx\right] = 0$$

Substituting $y = f\left(1-\frac{x^2}{a^2}\right)$

$$\int_{-a}^{a}\left[-H_4 f^2\left(1-\frac{x^2}{a^2}\right)^2 + \frac{q a^2 f^2}{6R}\left(1-\frac{x^2}{a^2}\right)^2 - \frac{q a^2 f^3}{12Rf}\left(1-\frac{x^2}{a^2}\right)^3\right] dx = 0$$

$$\int_{-a}^{a}\left[-H_4 f^2\left(1-\frac{2x^2}{a^2}+\frac{x^4}{a^4}\right) + \frac{q a^2 f^2}{6R}\left(1-2\frac{x^2}{a^2}+\frac{x^4}{a^4}\right) - \frac{q a^2 f^2}{12R}\left(1-3\frac{x^2}{a^2}+3\frac{x^4}{a^4}-\frac{x^6}{a^6}\right)\right] dx = 0$$

Intergrating $\left[-H_4 f^2\left(x-\frac{2x^3}{3a^2}+\frac{x^5}{5a^4}\right) + \frac{q a^2 f^2}{6R}\left(x-\frac{2x^3}{3a^2}+\frac{x^5}{5a^4}\right) - \frac{q a^2 f^2}{12R}\left(x-\frac{3x^3}{3a^2}+\frac{3x^5}{5a^4}-\frac{x^7}{7a^6}\right)\right]_{-a}^{a} = 0$

$$= 2\left[-H_4 f^2 a\left(1-\frac{2}{3}+\frac{1}{5}\right) + \frac{q a^3 f^2}{6R}\left(1-\frac{2}{3}+\frac{1}{5}\right) - \frac{q a^3 f^2}{12R}\left(1-1+\frac{3}{5}-\frac{1}{7}\right)\right] = 0$$

$$= \frac{-H_4 f^2 a(8)}{15} + \frac{q a^3 f^2}{12R}\left(\frac{8\times 2}{15} - \frac{16}{35}\right) = 0$$

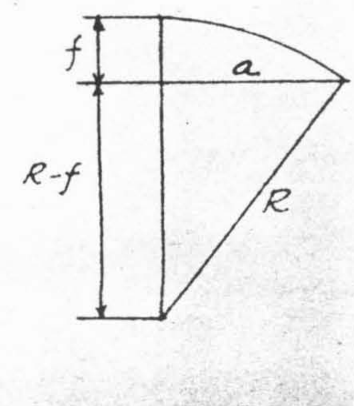
$$\therefore H_4 = \frac{q a^3 f^2}{12Raf^2}\left(\frac{16\times 7 - 16\times 3}{105}\right)\frac{15}{8} = \frac{q a^2}{R}\left(\frac{16\times 4}{12\times 7\times 8}\right) = \frac{2}{21}\frac{q a^2}{R}$$

But $f \cong \frac{a^2}{2R}$ This Approximation is Obtained As Follows:-

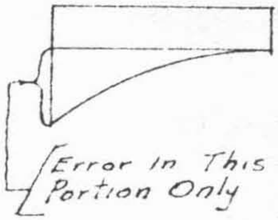
From The Triangle with sides $R, a, (R-f)$

$a^2 + (R-f)^2 = R^2$ or $a^2 = R^2 - R^2 + 2Rf - f^2$

$2Rf = a^2 + f^2$ or $f = \frac{a^2}{2R} + \frac{f^2}{2R}$



Since a^2 is Large Compared with f^2 And Effect only a Small Proportion of The Dead Load The Second Term on The Right Side is Neglected



The Percentage of Error is Less Than .005% of The H Value of The Dead Load Only.

Placing $f = \frac{a^2}{2R}$ In $H_4 = \frac{2}{21}\frac{q\,a^2}{R}$

$$H_4 = \frac{4}{21} q f \qquad \boxed{4B}$$

For The Uniform Load q, The Formulas For Live Load May Be Used. The Horizontal Reaction From This Portion of The Load is:-

$$H_3 = qR - \frac{4}{7} q f \quad (\text{See Eq } 3)$$

Total H for Dead Load is $H_D = H_3 + H_4 = qR - \frac{8}{21} q f$ (4)

Moments

The Moment Produced By The Uniform Portion of Dead Load is From Eq $\boxed{2}$

$$M_3 = \frac{q f^2}{14}\left(1 - \frac{8x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

The Moment Produced By The Additional Load is As Follows:-

From Eq $\boxed{4A}$ $M_4 = -H_4 y + \frac{q f a^2}{12 R}\left(1 - \frac{x^4}{a^4}\right)$

Eq $\boxed{4B}$ $H_4 = \frac{4}{21} q f$

$$\therefore M_4 = -\frac{4}{21} q f y + \frac{q f a^2}{12 R}\left(1 - \frac{x^4}{a^4}\right)$$

Substitute $f = \frac{a^2}{2R}$ $M_4 = -\frac{4}{21} q f y + \frac{q f^2}{6}\left(1 - \frac{x^4}{a^4}\right)$ By $y = f\left(1 - \frac{x^2}{a^2}\right)$

Substitute $y = f\left(1 - \frac{x^2}{a^2}\right)$ $M_4 = -\frac{8}{42} q f^2\left(1 - \frac{x^2}{a^2}\right) + \frac{7}{42} q f^2\left(1 - \frac{x^4}{a^4}\right)$

$$\therefore M_4 = -\frac{q f^2}{42}\left(8 - \frac{8x^2}{a^2} - 7 + \frac{7x^4}{a^4}\right) = -\frac{q f^2}{42}\left(1 - \frac{8x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

The Total Moment is Therefore $M_D = M_3 + M_4$

$$M_D = \frac{q f^2}{21}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right) \quad \boxed{5}$$

It is Noted That for Moment The Dead Load is Equivalent To $\frac{2}{3}$ of The Same Load Considered As A Live Load. Therefore For Moment Calculation The Dead Load Can Be Considered a Proportional Part of Live Load P (See Eq. $\boxed{2}$ & Eq $\boxed{5}$)

Combined Live & Dead Load

$$H = H_l + H_d = (P+g)R - \left(\frac{4}{7}P + \frac{8}{21}g\right)f$$

$$M = M_2 + M_D = \frac{1}{14}\left(P + \frac{2}{3}g\right)f^2\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right) \quad [6]$$

To Determine Points of Maximum Positive And Negative Moments Equate The First Derivative of [6] To zero.

$$\text{or } \frac{d}{dx}\left[\frac{Pf^2}{14}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right) + \frac{gf^2}{21}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right)\right] = 0$$

One Value is when $x = 0$ Placing This Value in [6]

$$M = \frac{f^2}{14}\left(P + \frac{2}{3}g\right) \quad [7]$$

Differentiating

$$-\frac{Pf^2}{14} \times \frac{(8)(2)x}{a^2} + \frac{Pf^2(4)x^3}{a^4} - \frac{gf^2}{21}\frac{(8)(2)x}{a^2} + \frac{gf^2}{3}\frac{4x^3}{a^4} = 0$$

Divide Thru By X

$$-\left(\frac{Pf^2}{14}\cdot\frac{16}{a^2}\right) + \left(\frac{Pf^2}{2}\cdot\frac{4x^2}{a^4}\right) - \left(\frac{gf^2}{21}\cdot\frac{16}{a^2}\right) + \left(\frac{gf^2}{3}\cdot\frac{4x^2}{a^4}\right) = 0$$

$$\frac{4x^2}{a^4}\left(\frac{Pf^2}{2} + \frac{gf^2}{3}\right) - \frac{16}{a^2}\left(-\frac{Pf^2}{14} + \frac{gf^2}{21}\right) = 0$$

$$\frac{4x^2}{a^4}\left(\frac{21Pf^2}{42} + \frac{14gf^2}{42}\right) - \frac{16}{a^2}\left(\frac{3Pf^2}{42} + \frac{2gf^2}{42}\right) = 0$$

$$4x^2 = \frac{16a^4}{a^2}\,\frac{\frac{3Pf^2}{42} + \frac{2gf^2}{42}}{\frac{21Pf^2}{42} + \frac{14gf^2}{42}} \quad \text{or } X^2 = 4a^2\left(\frac{\frac{1}{42}\left[3Pf^2 + 2gf^2\right]}{\frac{1}{42}\left[21Pf^2 + 14gf^2\right]}\right)$$

$$X^2 = \frac{4a^2}{7} \quad \text{or } X = a\sqrt{\frac{4}{7}}$$

Substitute This Value of X in [6]

$$M = \frac{Pf^2}{14}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right) + \frac{gf^2}{21}\left(1 - \frac{8x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

$$= \frac{Pf^2}{14}\left[1 - 8\left(\frac{4}{7}\right) + 7\left(\frac{4^2}{7^2}\right)\right] + \frac{gf^2}{21}\left[1 - 8\left(\frac{4}{7}\right) + 7\left(\frac{4^2}{7^2}\right)\right]$$

$$= \frac{9f^2}{7\times14}\left(P + \frac{2}{3}g\right) = \frac{9}{98}f^2\left(P + \frac{2}{3}g\right) \quad [8]$$

THRUST - DEAD LOAD

The Thrust At Any Arbitrary Point is Equal To

$$T_d' = \sqrt{H_d^2 + V_{xd}^2} \qquad V_x = qx + \int_0^x q \frac{f}{R} \times \frac{x^2}{a^2} dx'$$

$$\text{or } V_x = qx + q\frac{f}{R} \times \frac{x^3}{3a^2}$$

$$T_d = \sqrt{H_d^2 + q^2\left(x + \frac{f}{R}\frac{x^3}{3a^2}\right)^2}$$

THE THRUST AT THE SILLS

Substitute in The Above Equation The Value $qR - \frac{8}{21}qf$ For H_d

Also $a = x$

$$T_o = q\sqrt{\left(R - \frac{8}{21}f\right)^2 + \left(a + \frac{f}{R}\frac{a^3}{3a^2}\right)^2}$$

$$\text{or } T_o = q\sqrt{\left(R - \frac{8}{21}f\right)^2 + a^2\left(1 + \frac{f}{3R}\right)^2} \qquad \text{As Before } a^2 = 2Rf$$

$$= q\sqrt{R^2 - \frac{16}{21}Rf + \frac{64}{441}f^2 + 2Rf\left[1 + \frac{2f}{3R} + \frac{f^2}{9R^2}\right]}$$

$$= q\sqrt{R^2 - \frac{16}{21}Rf + \frac{64}{441}f^2 + 2Rf + \frac{4Rf^2}{3R} + \frac{2Rf^3}{9R^2}}$$

$$= q\sqrt{R^2 + \frac{26}{21}Rf + \frac{652}{441}f^2 + \frac{2f^3}{9R}}$$

$$= q\sqrt{R^2 + \frac{26}{21}Rf + \frac{169}{441}f^2 + \frac{483}{441}f^2 + \frac{2f^3}{9R}}$$

$$= q\sqrt{\left(R + \frac{13}{21}f\right)^2 + \frac{483}{441}f^2 + \frac{2f^3}{9R}}$$

The Last Two Terms Are Neglible In Comparison With The First Term So May Be Neglected (Approx 1%)

$$\therefore T_o = q\left(R + \frac{13}{21}f\right)$$

THRUST - COMBINED

The Total Maximum Thrust At The Sill is With Good Approximation:-

$$T_o\ \text{LIVE} = PR + P\frac{3}{7}f$$

$$T_o\ \text{DEAD} = qR + \frac{13}{21}qf$$

$$T\ \text{Combined} = R(P+q) + f\left(\frac{3}{7}P + \frac{13}{21}q\right)$$

The Minimum Value of The Thrust At The Crown Amounts To:-

$$T\ \text{min.} = H\ \text{Combined} = (P+q)R - \left(\frac{4}{7}P + \frac{8}{21}q\right)f$$

Summary

Of Formulas Obtained From The Preceeding Analysis

$$H = (P+q)R - \left(\frac{4}{7}P + \frac{8}{21}q\right)f$$

$$M_X = \frac{f^2}{14}\left(P+\frac{2}{3}q\right)\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

$$M_{MAX+} = \frac{f^2}{14}\left(P+\frac{2}{3}q\right)$$

$$M_{MAX-} = -\frac{9}{98}f^2\left(P+\frac{2}{3}q\right)$$

$$T_{COMBINED} = R(P+q) + f\left(\frac{3}{7}P + \frac{13}{21}q\right)$$

$$T_{MINIMUM} = H_{COMBINED} = (P+q)R - \left(\frac{4}{7}P + \frac{8}{21}q\right)f$$

Comparing The Results Obtained By Use of These Formulas And Those Obtained By The Exact Method for a Particular Case Follows:-

	Recommended Formulas	Exact Formulas
H	1584.2 Lbs	1584.0 Lbs
M_P	244.0 Ft Lbs	249.0 Ft Lbs
M_N	316.0 Ft Lbs	317.0 Ft Lbs

Denotations

L	Span of roof
a	Half span of arches
R	Radius of Arch
D	Distance between ends of building
f	Rise of roof
ℓ	Length of Lamellas
e	Eccentricity of joints
b	Width of Lamellas
h	Height of Lamellas
B	Inclination between Lamellas
s	Width of Diamonds
ϕ_c	Half central angle of arches
A_s	Cross-section area of sills
A_L	Cross-section area of Lamellas
E	Young's modulus for lumber
p	Live load / sq. foot of Horizontal Projection
q	Dead load / sq. foot of Length of Arc
w	Wind load / sq. foot of Vertical Projection
x	Ordinate from center of chord to point under consideration
y	Abcissa from the chord to the point under consideration

II. Analysis for Horizontal Load

1. Analysis of Arches for Wind Load

In order to compute additional stresses due to wind load, uniformly distributed horizontal forces of the magnitude w per unit vertical projection area are assumed, acting on one side of the roof. Reactions and moments are calculated using a similar method as under I. 1 and I. 2. The vertical reactions V_A and V_B are given by the equilibrium conditions

$$V_A = -V_B$$

$$V_A\, L = -\frac{wf^2}{2}$$

or $$V_A = -\frac{wf^2}{4a}$$

Denoting the horizontal reactions by H_A and H_B, the moment can be written $M = V_A\,(x + a) - H_A\, y - \frac{wy^2}{2}$ for $x < 0$

and $$M = V_A\,(x + a) - H_A\, y - wf(y - f/2) \text{ for } x > 0$$

Putting again approximately $y = f(1 - \frac{x^2}{a^2})$

and calculating

$$\int_{-a}^{a} My\, dx = 0$$

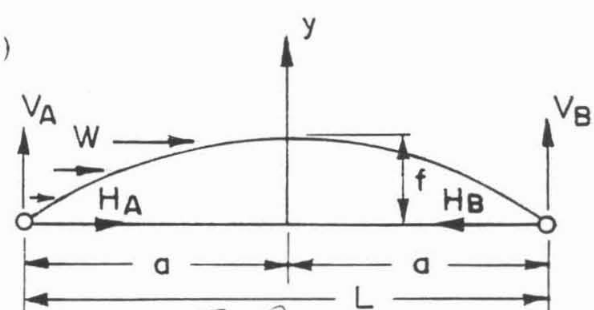


Fig 8

it is obtained $H_A = -\, 5/7\ wf$

and $H_B = +\, 2/7\ wf$

The negative sign of H_A indicates that the end thrust at the windside is diminished by $-\, 5/7$ wf. The horizontal reaction due to the weight of the roof is equal to $qR - \frac{8}{21}\, qf$. In general $qR - \frac{8}{21}\, qf \gg 5/7\ wf$. At the leeside the thrust is increased by $H_B = 2/7$ wf and the vertical reaction is $V_B = \frac{wf^2}{4a}$. The moment formula can be written with

$$H_A = -\, 5/7\ wf$$

$$V_A = -\frac{wf^2}{4a}$$

$$M_w = \frac{wf^2}{28}\left\{-1 - 7\frac{x}{a} + 8\left(\frac{x}{a}\right)^2 - 14\left(\frac{x}{a}\right)^4\right\} \text{ for } x < 0$$

$$M_w = \frac{wf^2}{28}\left\{-1 - 7\frac{x}{a} + 8\left(\frac{x}{a}\right)^2\right\} \text{ for } x > 0$$

The moment M_w is to be added M_1 and M_d and the maximum value to be determined from the plotted resulting curve.

2. Combined Wind and Earthquake Load

The foregoing analysis is based on the assumption that the sill does not undergo any horizontal displacement. In order to calculate the stresses in the joints and the deformation of the sill in the worst case, it is necessary to consider the whole roof as subjected to horizontal forces acting on the sills. As for the magnitude of these forces, we assume

or a) the total wind load acting on the forward sill

b) the weight of the roof and that part of the walls effective at the roof line multiplied by the seismic factor and divided equally between the two sills.

The greater of these two values is used in the design. The horizontal forces per unit length of the roof will be denoted by H. It is assumed that the roof can be considered as a trussed beam, and in the following analysis the maximum stresses occurring in the joints and the total horizontal deflection of the roof are calculated.

3. Maximum Stresses in Joints and Sills

In order to evaluate the maximum stresses occurring in the joints, the conditions near to the end of the building are to be considered. The total shearing force across the roof is equal to $\frac{HD}{2}$

(D distance between ends of the building). The distribution of the shearing forces along the end arch is assumed to follow the same parabolic law as the shearing stress in the cross section of the beam. Hence the maximum shear for unit length is $\frac{3}{2}\,\frac{HD}{2L}$ (L =span). If the number of diamonds along the end arch is n, the maximum value of the thrust or tension occurring in any of the end lamellas is equal to $\frac{L}{2n} \cdot \frac{3}{2} \cdot \frac{HD}{2L} = \frac{3}{8n} \cdot \frac{D}{L} \cdot H$. (As this maximum value occurs at the highest point of the barrel, no correction for curvature is necessary). In order to obtain the maximum possible value of compression, this quantity is to be added to the maximum compression due to live and dead load. In order to obtain the greatest possible value of tension, the difference between $\frac{3L}{8n} \cdot \frac{D}{L} \cdot H = \frac{3DH}{8n}$ and the minimum value of the thrust due to dead load only is to be calculated. The calculations of the joints, according to 5a), 5b), 5c), are to be repeated using these values for the thrust.

Furthermore, the stress occurring in the sills and calculated under I. 4 is to be corrected by additional stress due to wind or earthquake loading. The effect of the sheathing will be neglected in this calculation so that it is assumed that the total bending moment is carried by the sills. In this case the roof is equivalent to a beam with the section modulus ($A_S \cdot L$), (where A_S is the cross-section of one sill and L is the span). The stress due to the lateral load is therefore $\sigma = \frac{HD^2}{8L \cdot A_S}$.

HORIZONTAL DEFLECTION OF THE LAMELLA ROOF UNDER LATERAL LOADING

The roof is considered as a truss beam in the first approximation; it will be shown that the influence of the curvature is small as long as the tie rods are under tension and the two sills undergo approximately equal deflection. Due to the fact that the "height" of the beam, i.e., the span of the roof is nearly the same magnitude as its length, that is the length of the building, the "shear deflection" - which in the case of ordinary long beams is only a comparatively small portion of the total deflection - in this case is larger than the contribution of the pure bending. The main problem is, therefore, to determine the stiffness of the truss against shear.

It is assumed that uniformly distributed forces of the magnitude "H" lbs. per unit length are acting on the roof in the horizontal direction. The total deflection can be estimated as the sum of the following contributions:

(a) DEFLECTION due to Bending in the Horizontal Plane

The roof is considered as a beam with the inertial moment $I = \frac{L^2 A_s}{2}$. The deflection due to bending, i.e., to the extension or compression of the sills (E = Young's modulus) is given by $f_1 = \frac{5}{384} \frac{H D^4}{EI}$. Substituting the value $\frac{L^2 H_s}{2}$ for I we get

$$f_1 = \frac{5}{384} \frac{H D^4}{EI} = \frac{5}{192} \frac{H}{E} \frac{D^4}{L^2 H_s}$$

(b) DEFLECTION due to shear

Consider a row of diamonds (Fig. 8). The shearing force is denoted by S for this calculation it is permissible to distribute the force S uniformly over the diamonds. The average shearing force per diamond is equal to $\frac{S}{n}$ the tension or compression in the lamellas is $\frac{\pm S}{2n \cos \frac{B}{2}}$ (2 lamellas in each diamond).

Assuming that under the action of this force the length of the tension members is increased by $\Delta \ell$ and the length of the compression members is shortened by the same amount. The relative displacement of the opposite edges of the diamond will be equal to $\Delta \ell \cos \frac{B}{2}$. The distance of the edges being equal to $\mathcal{L}$ the relative displacement of two cross-sections of the roof in unit distance will be equal to $\frac{\Delta \ell}{\mathcal{L}} \cos \frac{B}{2}$. Denoting the shear deflection at an arbitrary distance x from the end of the building by f_2

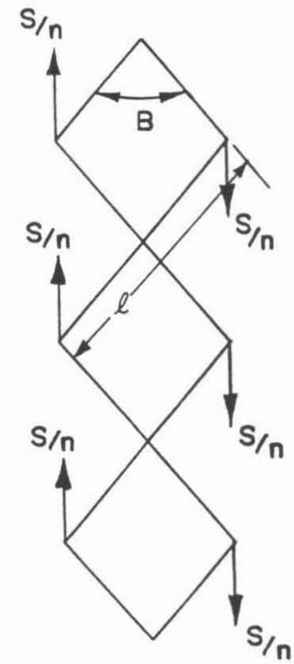


Fig 9

$$\frac{\Delta f_2}{dx} = \frac{\Delta \ell}{\mathcal{L}} \cos \frac{B}{2}$$

In order to carry out the calculation, it is necessary to compute $\Delta \ell$ i. e., the change of the length of the Lamellas.

$\Delta \ell$ is composed of the following:

(1) elastic extension and compression of the Lamellas

(2) bending of the lamellas

(3) relative displacement between the ends of the two spliced Lamellas connected with the same full Lamella - due to the eccentricity of the joints.

LET US ASSUME unit force acting as thrust in the Lamellas and calculate the three contributions (1), (2), and (3).

(1) <u>ELASTIC EXTENSION AND COMPRESSION OF THE LAMELLAS</u>

Denoting the average cross sectional area of the Lamella by A_L, the elastic modulus of the wood by E, the elastic change of the length ℓ of the Lamellas due to unit force is

$$\Delta \ell_1 = \frac{\ell}{EA_L}$$

(2) <u>BENDING OF THE LAMELLAS</u>

The Lamellas constitute skew arches. Due to the curvature of the arch the displacement $\Delta\ell$ is somewhat increased; however, this influence of the curvature is very small because the change in curvature of the arches subjected to tension is prevented by the arches subjected to compression and vice versa. If the number of diamonds between the sills is n every arch is divided into 2n sections, the end points of the sections being held by the system of arches intersecting with the arch considered. However, each such sector represents an arch and the bending of such an arch contributes a certain amount to $\Delta\ell$. The length of a sector is equal to half of the length of a Lamella or $\frac{\ell}{2}$, and the rise of the arch is $\frac{f}{(2n)^2}$, where f is the rise of the roof barrel.

Consider an arch of the length 2a and the rise f' under the action of a force of unit magnitude acting in the direction of the chord. The equation of a flat arch is approximately $y = f'(1 - \frac{x^2}{a^2})$ as used frequently in this analysis. The moment is therefore $M = f'(1 - \frac{x^2}{a^2})$ and the relative deflection of the two hinges (I = inertia moment of a Lamella)

$$\delta = \frac{1}{EI}\int_{-a}^{a} M y dx = \frac{f'^2}{EI}\int_{-a}^{a}(1 - \frac{x^2}{a^2})^2 dx = \frac{16}{15} a \frac{f'^2}{EI}$$

The change of length per Lamella $\Delta\ell$ will be, where $a = \frac{\ell}{4}$, $f' = \frac{f}{(2n)^2}$,

Substituting these values for a and f' in the above.

$$\Delta\ell_2 = 2\delta = \frac{8}{15}\frac{\ell}{EI}\frac{f^2}{(2n)^4} = \frac{1}{30}\frac{f^2\ell}{n^4 EI}$$

In order to compare this contribution $\Delta\ell_2$ with $\Delta\ell_1$, put $I = A_L \frac{h^2}{12}$ (h = height of Lamella)

$$\Delta\ell_2 = \frac{\ell}{EA_L}\frac{2}{5n^4}\frac{f^2}{h^2}$$

As an example

$f = 120''$, $h = 6''$, $n = 9$

$\Delta\ell_2 \cong \frac{1}{40}\Delta\ell_1$

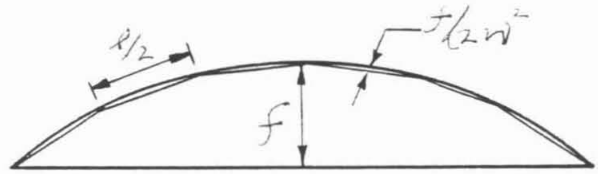


Fig 10

Contribution (2) is comparatively small.

(3) <u>INFLUENCE OF THE ECCENTRICITY OF THE JOINTS</u>

Due to the eccentricity of the joints the Lamellas are subjected to a bending moment. The deflection of the Lamella due to this moment increases the deflection of the roof in shear. The amount of this deflection is first determined for the Lamellas without any sheathing. The effect of the sheathing is then determined and the original defletion corrected, considering the restraint offered by the sheathing.

Consider the deflection of a full Lamella under the load transmitted by two spliced Lamellas. The thrust in the spliced Lamellas is assumed as unity (1), the eccentricity is denoted by e. The moment of the forces then equals e sin B. Let us consider the Lamella as a beam with free supported ends. The action of the two opposite forces of Magnitude T with lever arm e sin B can be replaced by a couple of the Magnitude Te sin B, acting at the center of the span ℓ (ℓ = length of a Lamella). The reactions at the two ends of the Lamella are equal to $R_1 = R_2 = \frac{e \sin B}{\ell}$.

The moment curve is represented by Figure 10. Calculating the deflection curve of the beam

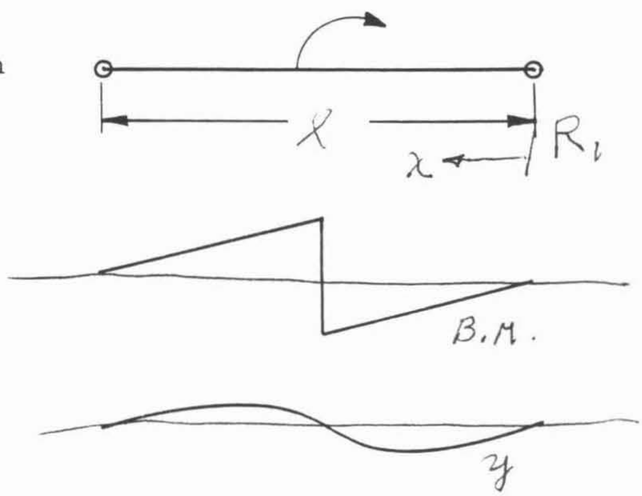


Fig 11

$$EI \frac{d^2 y}{dx^2} = -R_1 x$$

Integrating we obtain

$$y = -\frac{R_1}{EI}\frac{x^3}{6} + C_1 + C_2 x$$

(C_1 and C_2 are constants)

For $x = 0$ and $x = \frac{\ell}{2}$ $y = 0$

($x = 0$ supported end, for $x = \frac{\ell}{2}$ the deflection is zero for reasons of symmetry). Therefore $C_1 = 0$ and

$$-\frac{R_1}{EI}\frac{\ell^3}{48} + C_2 \frac{\ell}{2} = 0 \text{ , or } C_2 = \frac{R_1}{EI}\frac{\ell^2}{24}$$

(I = inertia moment of the Lamella cross section with respect to the longer axis = $\frac{hb^3}{12}$)

The equation of the deflection curve is therefore

$$y = \frac{R_1}{24EI} (x \, \ell^2 - 4x^3)$$

The slope of the beam, at an arbitrary point x, is given by

$$\frac{dy}{dx} = \frac{R_1}{24EI} (\ell^2 - 12x^2)$$

and the slope ϑ at the center of the beam, i. e., for $x = \frac{\ell}{2}$

$$\vartheta = \left(\frac{dy}{dx}\right)_{x = \frac{\ell}{2}} = - \frac{1}{12} \frac{R_1 \ell^2}{EI} = - \frac{1}{12} \frac{e \, \ell \sin B}{EI}$$

The relative displacement at the two joints perpendicular to the length axis of the Lamella is with good approximation equal to

$$\vartheta e = \frac{1}{12} \frac{e^2 \ell \sin B}{EI}$$

This deflection increases the elastic yielding of the shear members; the change in length per Lamella is equal to

$$\Delta \ell_3 = \vartheta e \sin B = \frac{1}{12} \frac{e^2 \ell \sin^2 B}{EI}$$

But $I = A_L \frac{b^2}{12}$ then

$$\Delta \ell_3 = \frac{\ell}{EA_L} \frac{e^2}{b^2} \sin^2 B$$

This value for $\Delta \ell_3$ is for the Lamellas alone without sheathing. We will now determine the effect of the sheathing on this value.

INFLUENCE OF THE SHEATHING ON THE DEFLECTION DUE TO THE ECCENTRICITY OF THE JOINTS

In this case of Horizontal loading, one system of arches, e.g., the system A, is subjected to compression, the other system, e.g., the 'B' arches - to tension. All bending moments applied at the

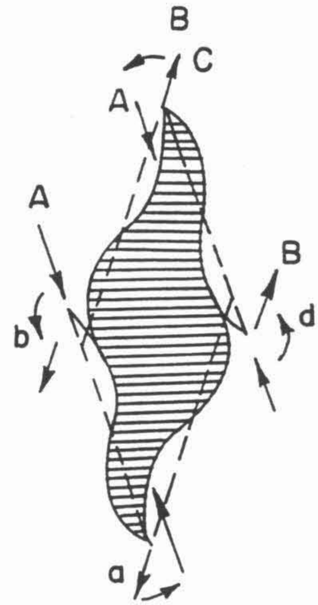


Fig 12

center of the Lamellas are acting anti-clockwise. Let us again consider two sides of a diamond meeting in the corner c. The half Lamellas bc and dc are connected by sheathing boards; consequently, bc and dc must have identical deflections. It is easy to see that this is only possible if both half-lamellas are deflected into an S shape. Thus every full Lamella is bent into a double S shape. Denoting the couple applied at the center of each Lamella by M = e sin B the moments applied by the sheathing at the ends of the Lamella amount to $-\frac{e \sin B}{2}$ and the corresponding moment curve is shown in Figure 13 as curve a). The formula for the moment is

$$M_{(x)} = -\frac{e \sin B}{2}\left(1 - \frac{4x}{\ell}\right)$$

The differential equation for the deflection.

$$IE\frac{d^2 y}{dx^2} = \frac{e \sin B}{2}\left(1 - \frac{4x}{\ell}\right)$$

Integrating this equation we obtain

$$y = \frac{e \sin B}{2IE}\left[\frac{x^2}{2} - \frac{2x^3}{3\ell} + Ax\right]$$

where A is a constant

For $x = \frac{\ell}{2}$, $y = 0$

Hence $A = -\frac{\ell}{12}$

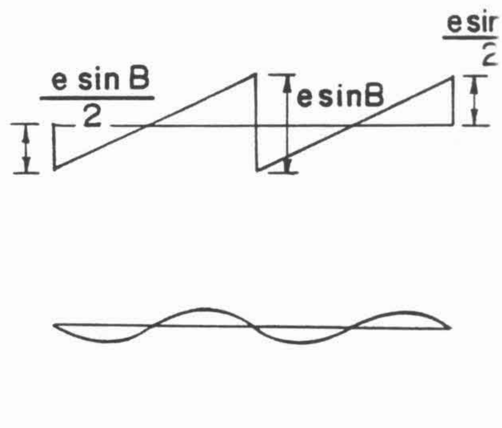


Fig 13

Then $y = \frac{e \sin B}{2EI}\left[\frac{x^2}{2} - \frac{2x^3}{3\ell} - \frac{\ell}{12}x\right]$

This curve is plotted in Fig. 13 as curve b).

The slope at the center is

$$\theta = \frac{dy}{dx} = -\frac{\ell e \sin B}{24IE}$$

and the relative displacement at the two joints perpendicular to the length axis of the Lamella is:

$$|\theta\epsilon| = \frac{1}{24}\,\frac{e^2 \ell \sin B}{IE}$$

Comparing this result given on Page H-9 we find that the deflection due to the eccentricity of the joints is reduced by the sheathing boards to one-half of the value calculated before.

A further reduction of the deflection is produced by the influence of the sheathing as a whole. Until now we have considered only the local stiffening effect of the sheathing boards. In addition to this effect the sheathing works as a whole, carrying a certain share of the moments produced by the eccentricity of the joints and reducing the bending moments acting on the Lamellas.

Figure 14 represents a section of the roof: the width of the section is equal to $\frac{\ell}{2}\cos\frac{B}{2}$ i.e., equal to one half of the longer diagonal of the diamonds. The length of the section considered is half of the length of the building $= \frac{D}{2}$. Due to the eccentricity of the joints,

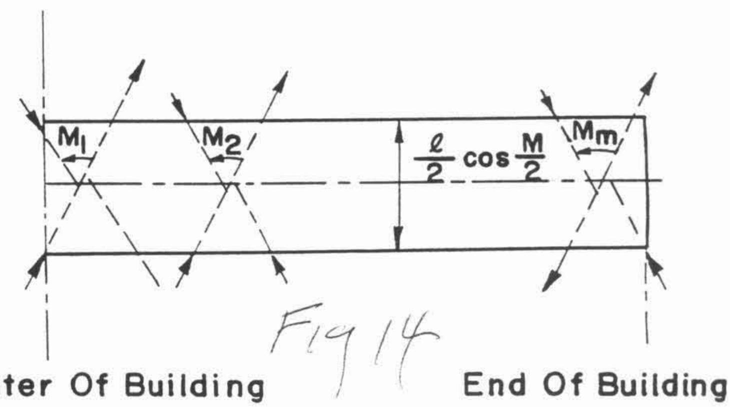


Fig 14

the couples, M_1, M_2 -- M_m are acting on the section considered.

These couples are approximately proportional to the distance from the center of the building, because the shear transferred at any cross-section of the roof, and therefore, also the thrust in the Lamellas, is proportional to this distance. Assuming that the sheathing is rigidly connected with the Lamellas, a certain portion of each couple will be carried by the sheathing and in this way the moments acting on the Lamellas will be reduced.

In order to calculate this reduction of the deflection, we first consider a simpler case.

The beam represented in Fig. 15 is subjected to the couple M acting at its center and supported elastically by boards similar to the sheathing boards of our roof. The elastic reaction q of the board per unit length of the beam is proportional to the deflection of the beam y. We write

$$q = -ky$$

Where k is a constant

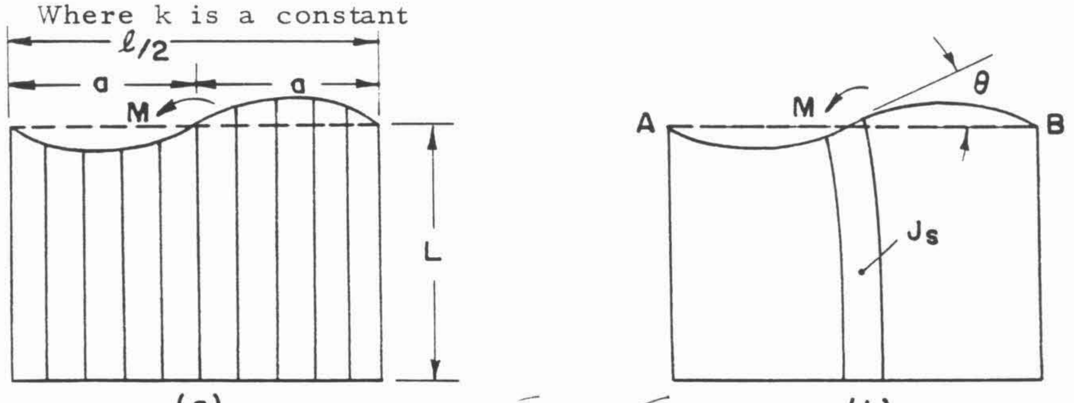


Fig 15

Let us calculate the reduction of the slope by the elastic reaction. The deflection y of the beam produced by the couple M is given by the differential equation:

$$IE \frac{d^2 y}{dx^2} = M(x) = \frac{M}{2} \frac{x}{a} \quad .$$

The solution of this equation - with the end conditions

$$y = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = a$$

is

$$y' = \frac{Ma^2}{12IE} \left(\frac{x}{a} - \frac{x^3}{a^3}\right)$$

The elastic reaction produced by this deflection is equal to

$$q = hy = \frac{Mka^2}{12IE} \left(\frac{x}{a} - \frac{x^3}{a^3}\right)$$

We first calculate the moment distribution produced by this load.

Denoting $-\frac{Mka^2}{12IE} = q_o$ the reaction A is given by the condition that the moment is zero at the center.

Hence $$Aa = - q_o \int_0^a \left(\frac{x}{a} - \frac{x^3}{a^3}\right)(a - x)\, dx$$

Integrating we get

$$A = - \frac{7}{60} q_o a$$

and the bending moment at an arbitrary point

$$M(x) = Ax + q_o \int_0^x \left(\frac{x'}{a} - \frac{x'^3}{a^3}\right)(x - x')\, dx'$$

Integrating, this becomes

$$M(x) = \frac{q_o a^2}{60} \left[- 7 \frac{x}{a} + 10 \left(\frac{x}{a}\right)^3 - 3\left(\frac{x}{a}\right)^5 \right]$$

The new deflection is given by

$$IE \frac{d^2 y}{dx^2} = M_{(x)}$$

or

$$IE \frac{d^2y}{dx^2} = \frac{q_o a^2}{60}\left[- 7 \frac{x}{a} + 10(\frac{x}{a})^3 - 3(\frac{x}{a})^5\right]$$

Integrating this equation and putting

$$y = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = a$$

we obtain finally

$$y = \frac{q_o a^4}{60EI}\frac{1}{42}\left[31 \frac{x}{a} - 49(\frac{x}{a})^3 + 21(\frac{x}{a})^5 - 3(\frac{x}{a})^7\right].$$

The slope for $x = a$ is equal to

$$\theta' = (\frac{dy}{dx})_{x = a}$$

substituting in the above equation.

$$\theta' = - \frac{4}{315}\frac{q_o a^3}{EI}$$

As previously determined the deflection due to a couple is

$$y = \frac{Ma^2}{12EI}\left[\frac{x}{a} - (\frac{x}{a})^3\right]$$

The slope is then

$$\theta = - \frac{Ma}{6EI}$$

The additional slope due to the elastic reaction is equal to θ' and with

$$q_o = - \frac{Mka^2}{12IE}$$

$$\frac{\theta'}{\theta} = - \frac{2}{315}\frac{ka^4}{IE}$$

To determine k. If the thickness of the sheathing is denoted by t, and its length by L, the stress is E y/L and the elastic reaction is equal to

$$q = -ky = -\frac{E\,y}{L}\,t$$

Hence

$$k = \frac{Et}{L}$$

and we obtain

$$\frac{\theta'}{\theta} = -\frac{2}{315}\frac{t}{L}\frac{a^4}{I}$$

We obtain the physical interpretation of this formula if we replace the sheathing by a beam of a certain flexural stiffness I_sE supporting the center of the beam. (Fig. 15b.) In this case we obtain a bending moment as an elastic reaction; its magnitude being $M' = I_sE\frac{\theta}{L}$ and as previously determined, the reduction of the slope of the Lamella at its center

$$\theta' = \frac{M'}{IE}\frac{a}{6}$$

Or

$$\theta' = -\frac{I_s}{I}\frac{a}{6L}\theta$$

Comparing the two formulas:

$$\frac{\theta'}{\theta} = -\frac{2}{315}\frac{ta^4}{LI}$$

and

$$\frac{\theta'}{\theta} = -\frac{1}{6}\frac{I_s}{I}\frac{a}{L}$$

we obtain the following expression for the equivalent flexural stiffness

$$I_s = \frac{4}{105}ta^3$$

We introduce the effective width of the sheathing putting

$$I_s = \frac{tw^3}{12}$$

Then

$$w^3 = \frac{16}{35} a^3$$

Or

$$w = 0.77a = 0.193\ell$$

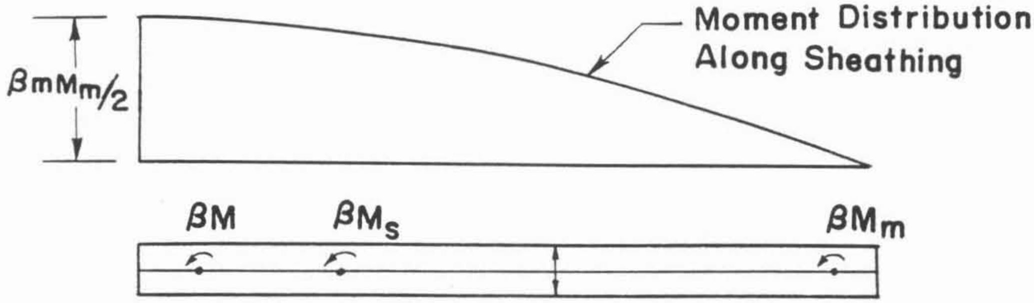
Since $a = \frac{\ell}{4}$ (Fig. 15a)

Effective width of sheathing is then 0.193 the length of a Lamella.

In the following calculations we use the effective or equivalent inertia moment of the sheathing obtained in this way, i.e., we replace the sheathing by a beam of the thickness t and height $w = 0.19\,\ell$ rigidly connected with the joints. In the actual computation we have to take into account that the Lamellas are not perpendicular to the sheathing. We take care of this effect by replacing

$$I_s \text{ by } I'_s = I_s \cos\frac{B}{2}$$

Denoting the couples acting at the joints by $M_1\ M_2\ \ldots\ M_m$ and assuming that M_m is the largest among the M, the total moment acting on the system will be $M_t = \frac{mM_m}{2}$. We assume furthermore that approximately the same portion of each couple is carried by the sheathing. We denote this portion by βM_1, $\beta M_2\ \ldots\ \beta M_m$. Then the total moment acting on the sheathing will be $\frac{\beta m M_m}{2}$. Replacing the couples βM_1, $\beta M_2\ \ldots\ \beta M_m$ by continuous loading the moment distribution of the bending moment along the sheathing or along the beam representing the sheathing will be parabolic, as shown in Fig. 16.



Hence, the slope of the beam at the end of the building will be

$$\theta = \int_0^{D/2} ds \, \frac{M_{(s)}}{I'_s E} = \frac{2}{3} \frac{D}{2} \frac{\beta m M_m}{2} \frac{1}{I'_s E}$$

(where D/2 = half the length of the building).

Setting this expression equal to the change of the slope of the last Lamella (at the end of the building). This change of slope was found to be equal to $- \frac{1}{6} \frac{Ma}{IE}$ where M is the couple acting at the center of the Lamella. Since (βM_m) is taken by the sheathing the net M acting on the lamella is $(1 - \beta) M_m$ and we write

$$\theta = \frac{(1 - \beta) M_m a}{6IE}$$

Comparing the two expressions for θ we find

$$(1 - \beta) \frac{M_m a}{6IE} = \frac{1}{6} \frac{\beta m M_m D}{I'_s E}$$

Or

$$\beta = \frac{1}{1 + m \frac{D}{a} \frac{I}{I'_s}}$$

and

$$1 - \beta = \frac{1}{1 + \frac{1}{m} \frac{a}{D} \frac{I'_s}{I}}$$

Since the slope of each Lamella at the joint is reduced in the ratio $(1 - \beta):1$ the quantity $1 - \beta$ will be the reduction factor for the deflection due to the eccentricity of the joints. The change in length $\Delta \ell$ was found, without considering the effect of the sheathing to be

$$\Delta \ell_3 = \frac{\ell}{EA_L} \frac{e^2}{b^2} \sin^2 B \quad .$$

The investigation of the local influence of the sheathing led to a reduction factor of 1/2, the investigation of the effect of the sheathing as a whole to a further reduction factor of

$$\frac{1}{1 + \frac{1}{m} \frac{a}{D} \frac{I'_s}{I}}$$

Therefore, the final contribution due to the eccentricity of the joints to the horizontal deflection will be

(where $a = \frac{\ell}{4}$; $I'_s = I$)

$$\Delta \ell_3 = \frac{\ell}{EA_L} \frac{e^2}{b^2} \sin^2 B \left(\frac{1}{2}\right) \left(\frac{1}{1 + \frac{1}{m} \frac{\ell}{4D} \frac{I'_s}{I} \cos B} \right)$$

The three contributions (1), (2) and (3) to $\Delta \ell$ are

$$\Delta \ell = \Delta \ell_1 + \Delta \ell_2 + \Delta \ell_3$$

Or

$$\Delta \ell = \frac{\ell}{EA_L} \left[1 + \frac{2}{5n^4} \frac{f^2}{h^2} + \frac{e^2}{2b^2} \sin^2 B \, \frac{1}{1 + \frac{1}{m} \frac{\ell}{4D} \frac{I_s \cos B}{I}} \right]$$

The $\Delta \ell$ thus obtained is the displacement from a force of unity. To determine the actual displacement we must determine the actual force acting on the Lamella. The resulting shearing force at a distance x from the center of the building is equal to Hx; however, to obtain the actual shear along the barrel, a correction for the curvature must be introduced. Assuming approximately uniform distribution of the shear over the barrel, Hx is to be multiplied with the mean value of $\frac{1}{\cos \varphi}$, where φ is the inclination of the elements of the arches toward the horizontal direction.

Putting approximately $\cos \varphi = 1 - \frac{\varphi^2}{2}$

$$\text{and} \quad \frac{1}{\cos \varphi} = 1 + \frac{\varphi^2}{2}$$

the mean value of $\frac{1}{\cos \varphi} = 1 + \frac{\varphi_o^2}{6}$

(where φ_o is half the central angle of the barrel in radians)

The mean value of the total force along the barrel is

$$Hx \left(1 + \frac{\varphi_o^2}{6}\right)$$

and the force in the direction of the Lamellas is

$$\frac{Hx \left(1 + \frac{\varphi_o^2}{6}\right)}{2n \cos \frac{B}{2}}$$

The displacement is then

$$\Delta \ell_{actual} = \frac{Hx \left(1 + \frac{\varphi_o^2}{6}\right)}{2n \cos \frac{B}{2}} \quad \frac{\ell}{EA_L} \left[1 + \frac{2}{5n^4} \frac{f^2}{h^2} + \frac{e^2}{2b^2} \sin^2 B \frac{1}{1 + \frac{1}{m} \frac{\ell}{4D} \frac{I_s \cos B}{I}} \right]$$

as previously determined

$$\frac{df_2}{dx} = \gamma = \frac{\Delta \ell \cos \frac{B}{2}}{\mathcal{L}} \quad \text{For 1 diamond}$$

or

$$\gamma = \frac{df_2}{dx} = \frac{Hx\left(1 + \frac{\varphi_0{}^2}{6}\right)}{2ns} \frac{\ell}{EA_L} \left[1 + \frac{2}{5n^4} \frac{f^2}{h^2} + \frac{e^2}{2b^2} \sin^2 B \frac{1}{1 + \frac{1}{m} \frac{\ell}{4D} \frac{I_s \cos B}{I}} \right]$$

For complete roof deflection $\frac{df_2}{dx} = n\gamma$

And with $\frac{s}{\ell} = \sin \frac{B}{2}$

$$n\gamma = \frac{df_2}{dx} = x \left[\frac{H}{2} \left(1 + \frac{\varphi_0{}^2}{6}\right) \frac{1}{\sin \frac{B}{2}} \frac{1}{EA_L} \right] \left[1 + \frac{2}{5n^4} \frac{f^2}{h^2} + \frac{e^2}{2b^2} \sin^2 B \frac{1}{1 + \frac{1}{m} \frac{\ell}{4D} \frac{I_s \cos B}{I}} \right]$$

The deflection due to shear is

$$f_2 = \int_0^{D/2} n\gamma \, dx$$

solving

$$f_2 = \frac{HD^2}{16} \left[\left(1 + \frac{\varphi_0{}^2}{6}\right) \frac{1}{\sin \frac{B}{2}} \frac{1}{EA_L} \right] \left[1 + \frac{2}{5n^4} \frac{f^2}{h^2} + \frac{e^2}{2b^2} \sin^2 B \frac{I}{1 + \frac{1}{m} \frac{\ell}{4D} \frac{I_s \cos B}{I}} \right]$$

The total deflection is, therefore, the deflection due to bending plus that due to shear or

$$f = f_1 + f_2$$

Therefore,

$$f = \frac{5}{192}\frac{H}{E_s}\frac{D^4}{L^2 A_s} + \frac{HD^2}{16}\left[\left(1+\frac{\varphi_0^2}{6}\right)\left(\frac{1}{\sin\frac{B}{2}}\frac{1}{EA_L}\right)\left(1+\frac{2}{5n^4}\frac{f^2}{h^2}+\frac{e^2}{2b^2}\sin^2 B\frac{1}{1+\frac{1}{m}\frac{\ell}{4D}\frac{I_s\cos B}{I}}\right)\right]$$

In this formula

H	denotes the horizontal force per foot length
D	length of the building
E	elastic modulus
φ_0	half center angle of arches
n	number of diamonds along the span
m	number of diamonds over half length of the building
e	eccentricity of joints
b	thickness of Lamellas
ℓ	length of Lamellas
B	angle between Lamellas
A_L	cross-section of Lamellas
I	inertia moment of Lamellas $= \frac{hb^3}{12}$
h	height of Lamellas
I_s	effective inertia moment of sheathing $\frac{(0.19\ell)^3\, t}{12}$
t	thickness of sheathing boards

ANALYSIS OF THE LAMELLA ROOF

VERTICAL FORCES

by

Dr. Theodor von Kármán

Director of Guggenheim Laboratory
California Institute of Technology.

external pressure). The corresponding figures for the vertical reaction V_ℓ and for the horizontal reaction H_ℓ due to this load, are the following:

$$V_1 = pR \sin \varphi_0$$

$$H_1 = pR \cos \varphi_0$$

Investigate the horizontal load. Denoting the contribution of this load to the horizontal reaction by H_2, the moment at an arbitrary point will be

$$M_2 = -H_2 y + \frac{p}{2} y^2$$

The equation for the determination of the statically undetermined quantity H_2 is $\int_{-a}^{+a} M y \, dx = 0$

Using the approximation $y = f\left(1 - \frac{x^2}{a^2}\right)$

we obtain

$$-H_2 f^2 \int_{-a}^{+a} \left(1 - \frac{x^2}{a^2}\right)^2 dx + \frac{pf^3}{2} \int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right)^3 dx = 0$$

or

$$H_2 = \frac{pf}{2} \frac{\int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right)^3 dx}{\int_{-a}^{a} \left(1 - \frac{x^2}{a^2}\right)^2 dx}$$

Fig. 1

and evaluating the integrals we get

$$H_2 = \frac{3}{7} pf$$

With this value of H_2 the moment M_2 will be

$$M_2 = -\frac{3}{7} pfy + \frac{1}{2} py^2$$

or introducing again $y = f\left(1 - \frac{x^2}{a^2}\right)$ and denoting the moment by M_ℓ (as due to the live load)

$$M_\ell = \frac{pf^2}{14}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

The total horizontal reaction due to the live load is obviously

$$H_\ell = H_1 + H_2 = pR \cos \varphi_0 + \frac{3}{7} pf$$

and with $\cos \varphi_0 = \frac{R - f}{R}$

$$H_\ell = pR - \frac{4}{7} pf$$

The thrust T_e at an arbitrary point x is

$$T_e = \sqrt{H^2 + (px)^2}$$

The thrust at the sills is equal to

$$T_0 = p\sqrt{(R - \frac{4}{7}f)^2 + a^2}$$

or approximately

$$T_0 = p(R + \frac{3}{7}f)$$

2. ANALYSIS OF THE ARCHES FOR DEAD LOAD

The dead load per unit length of the arc is denoted by q. The load per unit length of horizontal projection is $p_d = \frac{q}{\cos\phi}$. For $\phi = \pm\phi_0$ (at the hinges) $p_d = \frac{q}{\cos\phi_0} = \frac{qR}{R-f} \simeq q(1 + \frac{f}{R})$

Therefore, the dead load can be considered as a load q uniformly distributed over the horizontal projection plus an additional load. The additional load has its maximum value $q\frac{f}{R}$ at the hinges. The distribution of the additional load can be approximated by $q\frac{f}{R}\frac{x^2}{a^2}$

Consider half arch between $x = 0$ and $x = a$. The moment corresponding to the additional load amounts to

$$M_A = M_0 + H_A(f - y) - \frac{qf}{R}\frac{x^4}{12a^2}$$

where M_0 is the moment at $x = 0$

For $x = a$, $y = 0$, $M_A = 0$ therefore

$$M_0 + H_A f - \frac{qf}{R}\frac{a^2}{12} = 0$$

and

$$M_A = -H_A y + \frac{qf}{12}\frac{a^2}{R}\left(1 - \frac{x^4}{a^4}\right)$$

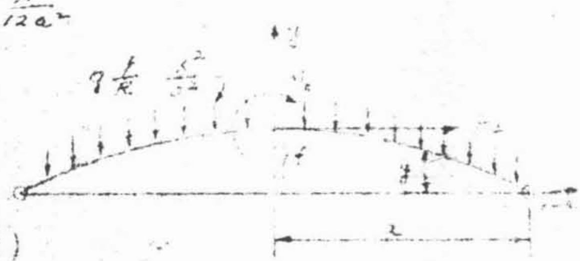


Fig. 2

Putting again $\int_{-a}^{a} M y \, dx = 0$

$$-H_4 f^2 \int_{-a}^{+a} \left(1-\frac{x^2}{a^2}\right)^2 dx + \frac{q f^2 a^2}{12R} \int_{-a}^{a} \left(1-\frac{x^4}{a^4}\right)\left(1-\frac{x^2}{a^2}\right) dx = 0$$

or

$$H_4 = \frac{q a^2}{12R} \frac{\int_{-a}^{+a} \left(1-\frac{x^4}{a^4}\right)\left(1-\frac{x^2}{a^2}\right) dx}{\int_{-a}^{+a} \left(1-\frac{x^2}{a^2}\right) dx}$$

Evaluating the integrals it is found:

$$H_4 = \frac{2}{21} q \frac{a^2}{R}$$

and because approximately $f = \frac{a^2}{2R}$ we get

$$H_4 = \frac{4}{21} q f$$

For the load q distributed uniformly per unit length of the horizontal projection, the formulas deduced for the live load can be used. Hence the horizontal reaction produced by the uniform load is

$$H_3 = qR - \frac{4}{7} q f$$

and the total horizontal reaction developed by the dead load

$$H_d = H_3 + H_4 = qR - \frac{8}{21} q f$$

The moment produced by the load uniformly distributed over the horizontal projection is given by

$$M_3 = -\frac{3}{7} q f y + \frac{1}{2} q y^2 \cong \frac{q f^2}{14}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

The moment produced by the additional load is

$$M_4 = -\frac{4}{21} q f y + \frac{q f^2}{6}\left(1 - \frac{x^4}{a^4}\right)$$

or

$$M_4 = \frac{-q f^2}{42}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

The total moment produced by the dead load is therefore

$$M_d = M_3 + M_4 = \frac{q f^2}{21}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

This result can be expressed as follows:

As far as the moment is concerned, the load distributed uniformly over the arc is equivalent to 2/3 of the same load distributed uniformly over the horizontal projection.

The thrust at an arbitrary point is equal to

$$T_d = \sqrt{H_d^2 + V_{xd}^2} \qquad V_x = q x + \int_0^x q \frac{f}{R} \frac{x'^2}{a^2} dx'$$

$$\text{or} \quad V_x' = q x + q \frac{f}{R} \frac{x^3}{3a^2}$$

$$T_d = \sqrt{H_d^2 + q^2 \left(x + \frac{1}{3} \frac{f}{R} \frac{x^3}{a^2}\right)^2}$$

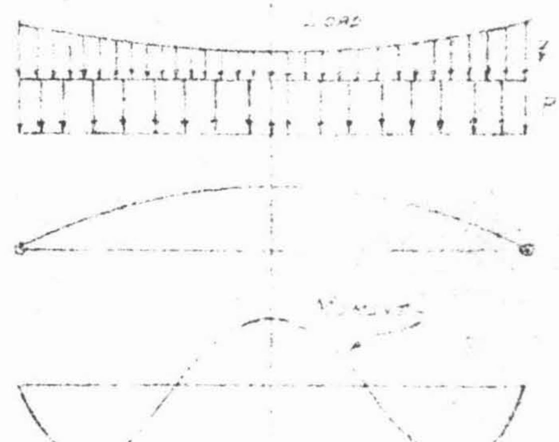


Fig. 3

The thrust at the sills is equal to

$$T_o = q \sqrt{\left(R - \frac{8}{21} f\right)^2 + a^2 \left(1 - \frac{f}{3R}\right)^2}$$

or approximately

$$T_o = q \left(R + \frac{13}{21} f\right)$$

3. FORMULAE FOR COMBINED LIVE AND DEAD LOAD

a) REACTIONS AND THRUST

The total vertical reaction is equal to

$$V = (p+q) a + \frac{2}{3} q \frac{f^2}{a}$$

the total horizontal reaction is

$$H = H_l + H_d = (p+q) R - \left(\frac{4}{7} p + \frac{8}{21} q\right) f$$

the total maximum thrust is with good approximation

$$T_{max} = (p+q) R + \left(\frac{3}{7} p + \frac{13}{21} q\right) f$$

and the minimum value of the thrust (at the peak of the arches) amounts to

$$T_{min} = H = (p+q) R - \left(\frac{4}{7} p + \frac{8}{21} q\right) f$$

Because of the comparatively slight variation of the thrust along the arch, the maximum values can be used for stress calculations.

b) MOMENT

The distribution of the resulting moment is given by the formula

$$M = M_c + M_a = \frac{1}{14}\left(p + \frac{2}{3}q\right) f^2 \left[1 - 8\left(\frac{x}{a}\right)^2 + 7\left(\frac{x}{a}\right)^4\right]$$

The positive maximum moment occurs at $x = 0$

$$M_p = \frac{1}{14}\left(p + \frac{2}{3}q\right) f^2$$

The negative maximum occurs at the point where

$$\frac{dM}{dx} = 0 \quad i.e. \quad -16\left(\frac{x}{a}\right) + 28\left(\frac{x}{a}\right)^3 = 0 \qquad or \quad \frac{x}{a} = \sqrt{\frac{4}{7}}$$

or $x = .76\,a$. The value of the negative maximum amounts to

$$M_n = -\frac{9}{98}\left(p + \frac{2}{3}q\right) f^2$$

It is seen that in absolute value $M_n > M_p$. Therefore, the maximum stress in the Lamellas can be written

$$\sigma_{max} = \left(\frac{T_{max}}{2A_l} + \frac{M_n}{S_l}\right) s$$

where A_l denotes the cross-section, S_l the section modulus of the Lamellas, and s is the spacing of the arches along the sill. The factor 2 in the denominator results from the fact that two arches carry the load corresponding to the spacing s.

c) INFLUENCE OF THE OBLIQUITY OF THE ARCHES

In order to check the error introduced by neglecting the inclination of the arches, the formulae for thrust and moment have to be compared with the exact calculation. The radius of the curvature of the inclined arches is equal to $\frac{R}{\cos B}$, on the other hand the load per unit length is $(p+q)\cos B$ so that the first term in the equation does not change; the second term is smaller in the case of the inclined arches

(f having the same value). Hence the approximate formula gives a somewhat larger value for the maximum thrust and is on the safe side. The same is true for the moment formula.

4. STRESSES IN TIE RODS AND SILLS

Denoting the spacing of tie rods with t, the spacing of the arches with s, the total tension in the tie rods will be

$$P_t = t\,(H_d + H_c)$$

With A_r cross-section area of tie rods the stress amounts to

$$\sigma = \frac{H_d + H_c}{A_r}\,t$$

and the extension

$$\Delta L = \frac{L}{E' A_r}\,\frac{(H_d + H_c)}{A_r}\,t$$

(E' = Young's modulus for steel = 30,000,000 lbs./sq.inch)

The amount $\frac{\Delta L}{2}$ gives the displacement of the sill due to the vertical load, if the friction between the sill and the walls is neglected. The magnitude of ΔL is to be compared with the deformation of the arch in the case no tie rods are provided. The magnitude of ΔL in this case is given by

$$\Delta L = \frac{1}{EI}\int_{-a}^{+a} M y \, dx$$

where M is the bending moment. In the case considered $H = 0$, $M = (p+g)\left(\frac{a^2 - x^2}{2}\right)s$

and with $y = f\left(1 - \frac{x^2}{a^2}\right)$

$$\Delta L = \frac{p+g}{EI}\,a^3\,\frac{fs}{2}\int_{-1}^{+1}\left(1-\left(\frac{x}{a}\right)^2\right)^2 d\left(\frac{x}{a}\right)$$

or
$$\Delta L = \frac{8}{15}\,\frac{p+q}{EI}\,a^3 f_s$$

This quantity is in all practical cases much larger than the elastic extension of the tie rods; hence the neglection of the elasticity of the tie rods, as far as the arch analysis is concerned, is justified.

The bending stress in the sills can be computed by the formula $\sigma = \frac{(H_d + H_e)\,t^2}{8 S_s}$, where S_s is the section modulus of the sill.

5. ANALYSIS OF JOINTS BETWEEN LAMELLAS AND BETWEEN LAMELLA AND SHEATHING BOARDS

a) JOINT BETWEEN FULL AND SPLICED LAMELLA

It is assumed that in the plane of the lamellas the following forces keep each other in equilibrium; (friction neglected)

Thrust in the cut Lamella = T

Bearing normal to full Lamella = Q

Tension in bolt = T_B

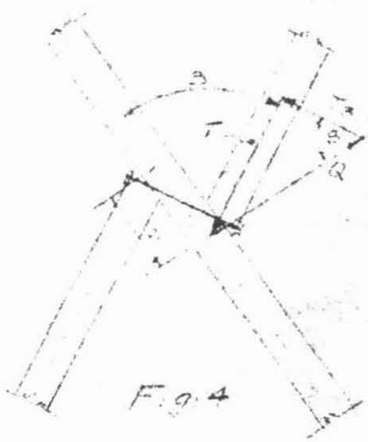


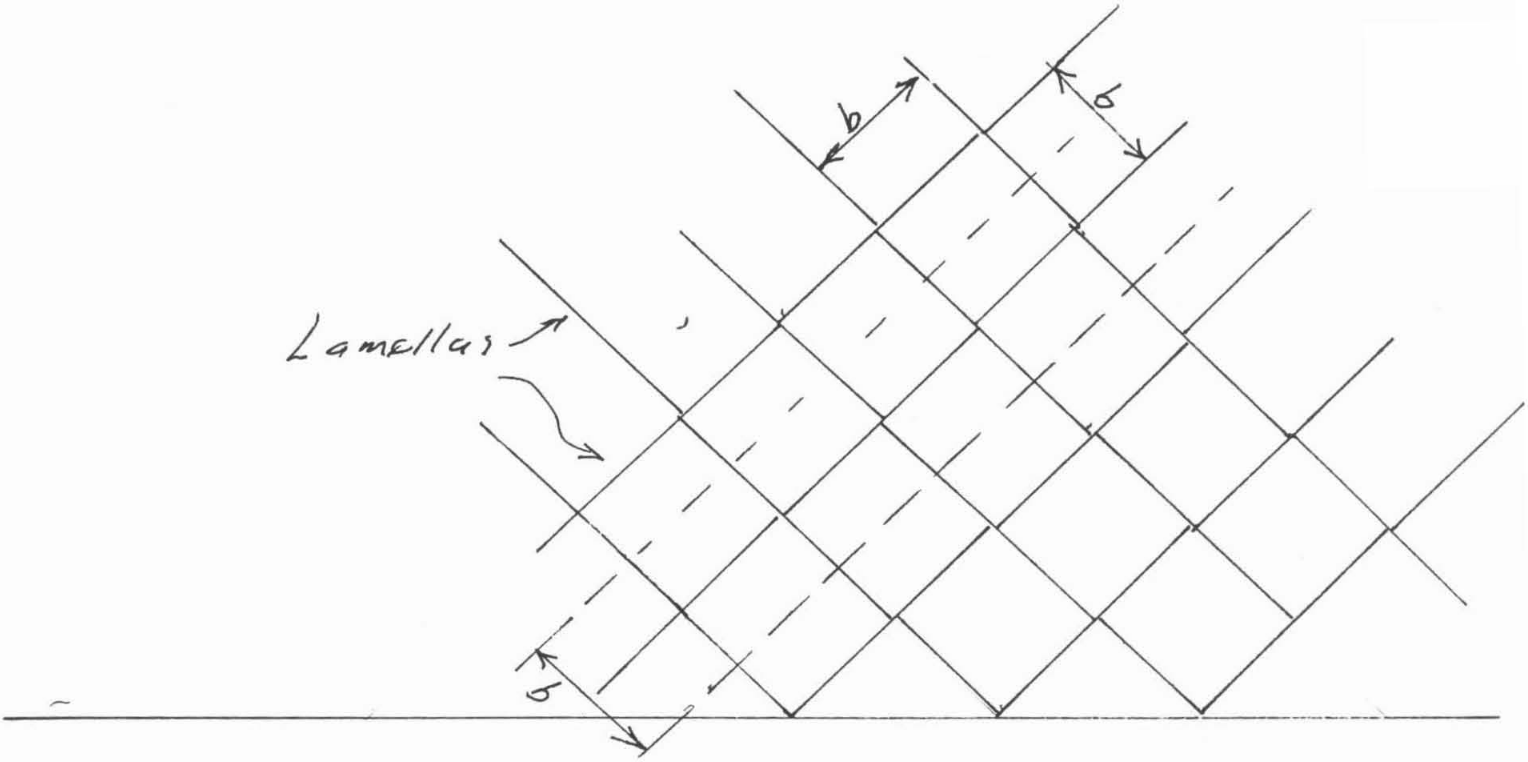
Fig. 4

Hence (Figure)
$$Q = T\,\frac{1}{\sin B}$$
$$T_B = T \cot B$$

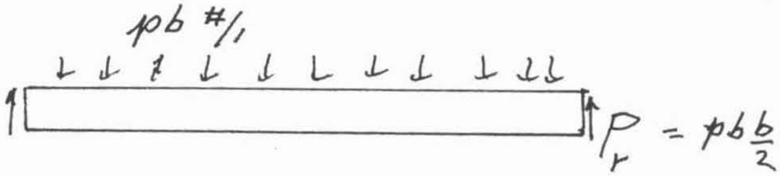
Bearing area = $\frac{\text{Minimum cross-section of Lamella}}{\sin B}$

Therefore

$$\text{Bearing stress} = \frac{\text{Thrust}}{\text{Minimum Cross-Section}}$$



Assume first that roof load p is carried just by one set of parallel lamellas



but the load p is carried equally by the two sets of lamellas, so take p/2

$$P_{r\,max} = p\,b^2/4$$

Memo to: Jean Anderson
From: George Housner
Subject: Lamella roof.

P_r was not calculated in the report, but it would have been the valus shown in the attached sheet.

G.H.

Refers to page V 9
line 6

Thrust = Thrust per unit length of roof x half spacing of arches.

Due to the normal or shearing force there is a tendency of gliding between the spliced and the full Lamella in a direction perpendicular to the tangential plane of the roof. The maximum shearing force P_r is calculated in . The friction coefficient necessary to prevent gliding is equal to $f = \frac{P_r}{\text{THRUST}}$ in general of the order $F = .05$.

b) INFLUENCE OF ECCENTRICITY OF JOINTS

The eccentricity of the joint connecting the two spliced Lamellas with the full Lamella produces a couple of the magnitude – $T e \sin B$ acting on the full Lamella. The sheathing is utilized to resist the moment.

It is assumed that the sheathing boards are rigidly connected with the Lamellas and their elastic extension can be neglected in comparison to the deflection of the Lamellas due to bending. Then obviously the half Lamellas, which constitute two sides of the diamonds, and are connected by sheathing boards, will undergo identical deflections.

In the case of vertical load both systems of arches are subjected to compression. Let us denote the arches running from the rear sill to the front sill from left to right the A system, the arches running from the front to the rear from left to right the B

Fig. 5

system. Both systems being subjected to compression, the A system imposes upon the Lamellas belonging to the B system bending moments acting anticlockwise, the B system applies upon the A-Lamellas bending moments acting clockwise. Without taking into account the sheathing boards, the bending moment curve a) and the deflection curve b) shown in Figure 6 were obtained. Due to the influence of the sheathing boards the deflections of the corresponding points on two sides of the diamond must be equal. Especially the angle B between the two sides connected by the sheathing boards cannot change. This condition involves the application of moments of the magnitude $\frac{Te}{2} \sin B$ at both ends of the Lamelle.

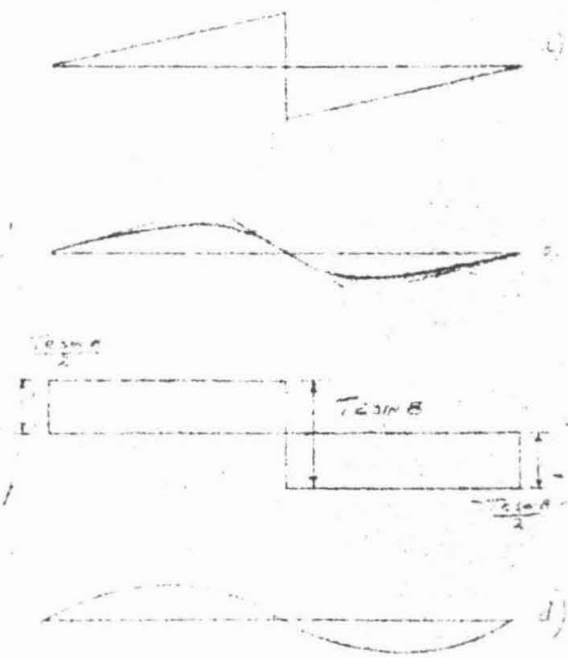


Fig 6

The moment curve obtained in this way is shown as c) in Figure 6. The slope of the deflection curve will be equal to the end and the center of the Lamella and amounts to $\theta = \frac{1}{8} \frac{Tel \sin B}{EI}$

The equation of the deflection curve is

$$y = \frac{1}{8} \frac{Tel \sin B}{EI} \left(x - \frac{2x^2}{l} \right)$$

The curve is plotted as d) in Figure 6

The nailing of the sheathing boards has to provide sufficient strength to resist the moment $\frac{Te}{2} \sin \beta$ Assuming that the moment is distributed upon three boards and denoting the width of each board by w, the shear to be carried by the nailing of one board will be

$$S = \frac{Te}{2} \sin \beta \left[\frac{1}{\frac{w}{2} + \frac{3w}{2} + \frac{5w}{2}} \right] = \frac{Te}{9w} \sin \beta$$

c) <u>JOINTS AT THE END OF THE BUILDING</u>

Because of the obliquity of the arches a lateral thrust is acting on the end arch of the building. The thrust per unit length is equal to $T \frac{\sin^2 \frac{\beta}{2}}{\cos \beta/2}$ The force at intersection of two arches is equal to $s\, T \sin \beta/2$. (T = Thrust per unit length of sill s = spacing). The spacing of the intersections along the end arch is approximately equal to

$$s \cot \beta/2$$

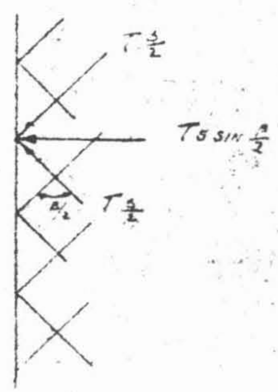
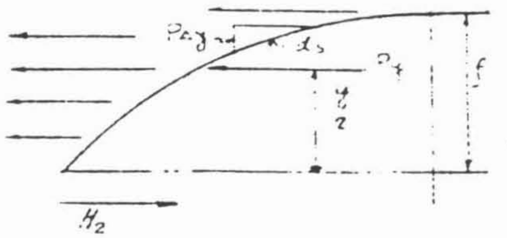


FIG. 7

Nails are to be provided with sufficient strength to transmit this force per unit length upon the sheathing.

DETAILED CALCULATIONS FOR VERTICAL LOADING

$M_2 = -H_2 y + P\frac{y^2}{2}$ $\boxed{A}$



For H_2 From The Formula =

$$\int_{-a}^{a} \frac{N m\, ds}{EI} = \delta$$

Since $\delta = 0$ (Rib Shortening Neglected)

M = Unity = y And $ds \cong dx$, E And I Are Constant This Becomes

$$\int_{-a}^{a} M y\, dx = 0 \quad \boxed{B}$$

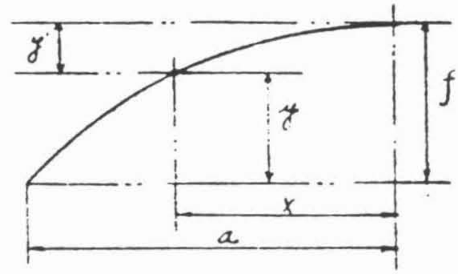
From The Parabola; $y = f(1-\frac{x^2}{a^2})$ Is Developed As Follows

Assume The Curve A Parabola With The Vertex At The Origin

$y' = kx^2$ (1)

When $x = a$, $y' = f$ $\therefore$ $f = ka^2$ (2)

$k = \frac{f}{a^2}$



Combine (1) And (2)

$y' = f\frac{x^2}{a^2}$

But $y = f - y' = f - f\frac{x^2}{a^2} = f(1-\frac{x^2}{a^2})$

Substitute Value of M_2 From $\boxed{A}$ in $\boxed{B}$

$$= \int_{-a}^{a}\left[-H_2 y^2 + \frac{Py^3}{2}\right] dx = 0$$

Substitute $y = f(1-\frac{x^2}{a^2})$

$$\int_{-a}^{a}\left[-H_2 f^2\left(1-\frac{x^2}{a^2}\right)^2 + Pf^3\frac{\left(1-\frac{x^2}{a^2}\right)^3}{2}\right] dx = 0$$

Expanding $\int_{-a}^{a}\left[-H_2 f^2\left(1 - 2\frac{x^2}{a^2} + \frac{x^4}{a^4}\right) + \frac{Pf^3}{2}\left(1 - \frac{3x^2}{a^2} + 3\frac{x^4}{a^4} - \frac{x^6}{a^2}\right)\right] dx = 0$

Intergrating $\left[-H_2 f^2\left(x - \frac{2x^3}{3a^2} + \frac{x^5}{5a^4}\right) + \frac{Pf^3}{2}\left(x - \frac{3x^3}{3a^2} + \frac{3x^5}{5a^4} - \frac{x^7}{7a^6}\right)\right]_{-a}^{+a} = 0$

Clearing $2\left[\frac{H_2 f^2 a(8)}{15} + Pf^3 a\frac{16}{70}\right] = 0$

Or $H_2 = \frac{3}{7}Pf$

Substituting in $\boxed{1A}$ $-M_2 = -H_2 y + \frac{py^2}{2}$

$$M_2 = -\frac{3}{7} pfy + \frac{1}{2} py^2 \quad \mathrm{I}$$

Substituting For y The Value $f(1 - \frac{x^2}{a^2})$

$$M_2 = -\frac{3}{7} pf^2\left(1 - \frac{x^2}{a^2}\right) + \frac{1}{2} pf^2\left(1 - \frac{x^2}{a^2}\right)^2$$

$$= -\frac{3}{7} pf^2 + \frac{\frac{3}{7} pf^2 x^2}{a^2} + \frac{1}{2} pf^2 - pf^2 \frac{x^2}{a^2} + \frac{1}{2} pf^2 \frac{x^4}{a^4}$$

$$= pf^2\left[-\frac{3}{7} + \frac{3}{7}\frac{x^2}{a^2} + \frac{1}{2} - \frac{x^2}{a^2} + \frac{x^4}{2a^4}\right]$$

$$= \frac{pf^2}{14}\left[-6 + 6\frac{x^2}{a^2} + 7 - \frac{14x^2}{a^2} + 7\frac{x^4}{a^4}\right]$$

$$\therefore M_2 = \frac{pf^2}{14}\left[1 - \frac{8x^2}{a^2} + 7\frac{x^4}{a^4}\right] \qquad \boxed{2}$$

The Total Horizontal Reaction For The Live Load

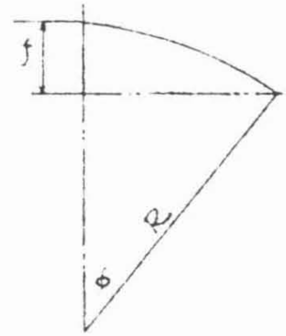
$$H_\ell = H_1 + H_2 = PR \cos\phi + \frac{3}{7} pf$$

$$\text{But} \quad \frac{R - f}{R} = \cos\phi$$

$$\therefore H_\ell = PR\,\frac{R-f}{R} + \frac{3}{7} pf$$

$$= \frac{PR^2 - PRf}{R} + \frac{3}{7} pf = PR - pf + \frac{3}{7} pf$$

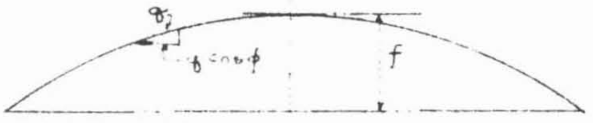
$$\therefore H_\ell = PR - \frac{4}{7} pf \qquad \boxed{3}$$

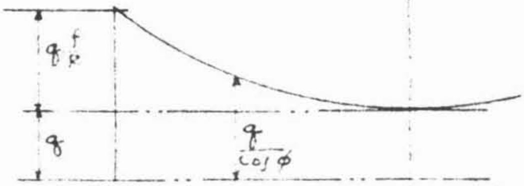


DEAD LOAD

Load per Unit Length of Horizontal Projection is $\frac{q}{\cos\phi}$

For $\phi = \pm\phi_0$ (At The Hinges) $\frac{q}{\cos\phi_0} = \frac{qR}{R-f} \simeq q(1 + \frac{f}{R})$





THRUST - LIVE LOAD:

The Thrust T_x At Any Arbitrary Point X is

$T_x = \sqrt{H^2 + V_x^2}$ $\qquad V_x = px$

The Thrust At The Sills. Substitute In The Above Equation The value $H_L = pR - \frac{4}{7}pf$ And for $x = a$

$T_o = p\sqrt{(R - \frac{4}{7}f)^2 + a^2}$

$T_o = p\sqrt{R^2 - 2R\frac{4}{7}f + \frac{16}{49}f^2 + a^2}$

Since $f = \frac{a^2}{2R}$; $a^2 = 2Rf$

Substituting

$T_o = p\sqrt{R^2 - \frac{8}{7}Rf + 2Rf + \frac{16}{49}f^2}$

$= p\sqrt{R^2 + \frac{6}{7}Rf + \frac{16}{49}f^2}$

$= p\sqrt{R^2 + \frac{6}{7}Rf + \frac{9}{49}f^2 + \frac{7}{49}f^2}$

$= p\sqrt{(R + \frac{3}{7}f)^2 + \frac{7}{49}f^2}$

$\frac{7}{49}f^2$ Is Neglible in Comparison With The First Term so May Be Neglected (Approx $\frac{1}{10}$ of 1%)

$\therefore T_o = p(R + \frac{3}{7}f)$

The Degree of Approximation is Shown Thus:

When $f = \frac{1}{10}R$ $\quad \frac{qR}{R-f} = q\,1.111$ And $q\left(1+\frac{f}{R}\right) = q\,1.1 = 99\%$ Exact.

$f = \frac{1}{5}R$ $\quad \frac{qR}{R-f} = q\,1.25$ And $q\left(1+\frac{f}{2}\right) = q\,1.2 = 96\%$ Exact.

$f = \frac{1}{6}R$ $\quad \frac{qR}{R-f} = q\,1.20$ And $q\left(1+\frac{f}{R}\right) = q\,1.167 = 97.5\%$ Exact.

This Percentage of Error, When The Total Live And Dead Load is Considered, For Roofs of $f = \frac{1}{6}R$ With a Thirty Pound Live Load is Approximately One Half of One Percent Error.

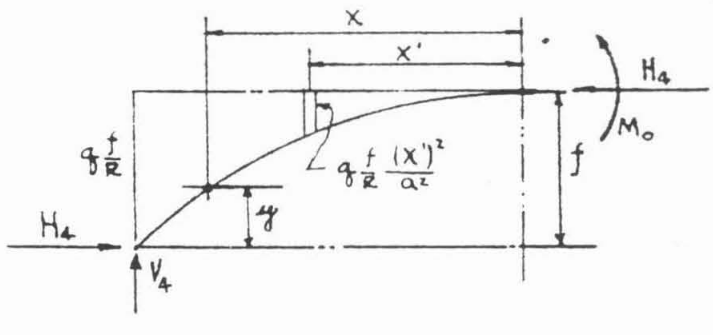
The Dead Load is Thus Assumed As A Load q Uniformly Distributed over The Horizontal Projection Plus An Additional Load. The Additional Load is $q \cdot \frac{f}{R} \cdot \frac{x^2}{a^2}$ The Maximum Value of $q \cdot \frac{f}{R}$ At The Hinges.

Considering The Half Arch Between $X = 0$ And $X = a$ The Moment From The Additional Load Becomes:

$$M_4 = M_o + H_4(f - y) - \int_0^x \left(q \frac{f}{R} \cdot \frac{x'^2}{a^2}\right)(x - x')\,dx'$$

Intergrating The Second Half of The Equation

$$\int_0^x \left[q \frac{f}{R} \frac{(x')^2}{a^2}(x - x')\right] dx'$$ Where X is The Constant, X' is The Variable



$$= \int_0^x \left[q \frac{f}{Ra^2}\left(x(x')^2 - (x')^3\right)\right] dx'$$

$$= \frac{qf}{Ra^2}\left[x\frac{(x')^3}{3} - \frac{(x')^4}{4}\right]_0^x$$

$$= \frac{qf}{Ra^2}\left[\frac{x^4}{3} - \frac{x^4}{4}\right] = \frac{qf\,x^4}{12Ra^2}$$

Hence ΣM To The Right of $A = M_o + H_4(f - y) - \frac{qf}{R}\frac{x^4}{12a^2}$

When $x = -a$: $y = 0$, $M_4 = 0$ $\quad \therefore M_4 = 0 = M_o + H_4 f - \frac{qf}{R} \cdot \frac{a^2}{12}$

And $M_o = -H_4 f - \frac{qf}{R} \cdot \frac{a^2}{12}$

Substitute This Value of M_o in The Equation For M_4

$$M_4 = -H_4 f + \frac{qf}{R}\frac{a^2}{12} + H_4 f - H_4 y - \frac{qf}{R}\frac{x^4}{12a^2}$$

$$= -H_4 y + \frac{qf}{R12}\left(a^2 - \frac{x^4}{a^4}\right)$$

$$\therefore M_4 = -H_4 y + \frac{qfa^2}{R12}\left(1 - \frac{x^4}{a^4}\right) \qquad [4A]$$

Again $\int_{-a}^{a} My\,dx = 0;\quad y = f\left(1-\frac{x^2}{a^2}\right);\quad \frac{ya^2}{f} = a^2 - x^2;\quad x^2 = a^2\left(1-\frac{y}{f}\right)$

Solving For H_4 Using The Above Value of M_4

$$M = -H_4 y + \frac{qfa^2}{12R}\left(1-\frac{x^4}{a^4}\right) = -H_4 y + \frac{qfa^2}{12R}\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{a^2}\right)$$

$$= -H_4 y + \frac{qa^2}{12R}\, y\left(1+\frac{x^2}{a^2}\right) = -H_4 y + \frac{qa^2}{12R}\, y\left[1+\left(1-\frac{y}{f}\right)\right]$$

$$= -H_4 y + \frac{qa^2}{6R}\, y - \frac{qa^2 y^2}{12R\,f}$$

$$\therefore \int_{-a}^{a} My\,dx = \int_{0}^{a}\left[-H_4 y^2\,dx + \frac{qa^2}{6R}\, y^2\,dx - \frac{qa^2}{12Rf}\, y^3\,dx\right] = 0$$

Substituting $y = f\left(1-\frac{x^2}{a^2}\right)$

$$\int_{-a}^{a}\left[-H_4 f^2\left(1-\frac{x^2}{a^2}\right)^2 + \frac{qa^2f^2}{6R}\left(1-\frac{x^2}{a^2}\right)^2 - \frac{qa^2f^3}{12Rf}\left(1-\frac{x^2}{a^2}\right)^3\right]dx = 0$$

$$\int_{-a}^{a}\left[-H_4 f^2\left(1-\frac{2x^2}{a^2}+\frac{x^4}{a^4}\right) + \frac{qa^2f^2}{6R}\left(1-2\frac{x^2}{a^2}+\frac{x^4}{a^4}\right) - \frac{qa^2f^2}{12R}\left(1-3\frac{x^2}{a^2}+3\frac{x^4}{a^4}-\frac{x^6}{a^6}\right)\right]dx = 0$$

Intergrating $\left[-H_4 f^2\left(x-\frac{2x^3}{3a^2}+\frac{x^5}{5a^4}\right) + \frac{qa^2f^2}{6R}\left(x-\frac{2x^3}{3a^2}+\frac{x^5}{5a^4}\right) - \frac{qa^2f^2}{12R}\left(x-\frac{3x^3}{3a^2}+\frac{3x^5}{5a^4}-\frac{x^7}{7a^6}\right)\right]_{-a}^{a} = 0$

$$= 2\left[-H_4 f^2 a\left(1-\frac{2}{3}+\frac{1}{5}\right) + \frac{qa^3f^2}{6R}\left(1-\frac{2}{3}+\frac{1}{5}\right) - \frac{qa^3f^2}{12R}\left(1-1+\frac{3}{5}-\frac{1}{7}\right)\right] = 0$$

$$= \frac{-H_4 f^2 a(8)}{15} + \frac{qa^3f^2}{12R}\left(\frac{8\times 2}{15} - \frac{16}{35}\right) = 0$$

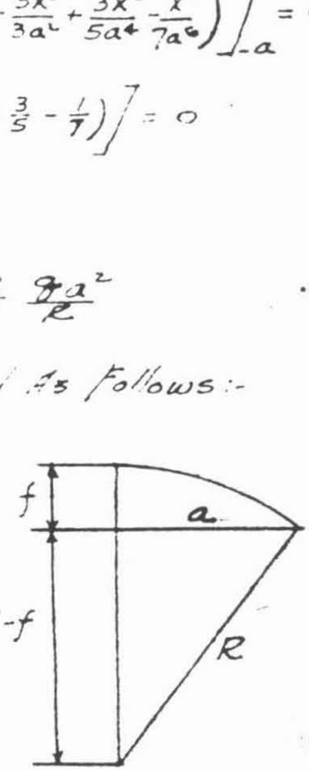
$$\therefore H_4 = \frac{qa^3f^2}{12Raf^2}\left(\frac{16\times 7-16\times 3}{105}\right)\frac{15}{8} = \frac{qa^2}{R}\left(\frac{16\times 4}{12\times 7\times 8}\right) = \frac{2}{21}\,\frac{qa^2}{R}$$

But $f \cong \frac{a^2}{2R}$ This Approximation is Obtained As Follows:-

From The Triangle with Sides R, a, $(R-f)$

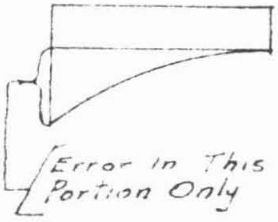
$a^2 + (R-f)^2 = R^2$ or $a^2 = R^2 - R^2 + 2Rf - f^2$

$2Rf = a^2 + f^2$ or $f = \frac{a^2}{2R} + \frac{f^2}{2R}$



Since a^2 is Large Compared with f^2 And Effect only a Small Proportion of The Dead Load The Second Term on The Right Side is Neglected

The Percentage of Error is Less Than .005% of The H Value of The Dead Load Only.



Placing $f = \frac{a^2}{2R}$ In $H_4 = \frac{2}{21}\frac{q a^2}{R}$

$$H_4 = \frac{4}{21} q f \qquad \boxed{4B}$$

For The Uniform Load q, The Formulas For Live Load May Be Used. The Horizontal Reaction From This Portion of The Load is:-

$$H_3 = qR - \frac{4}{7} q f \quad \text{(See Eq 3)}$$

Total H for Dead Load is $H_D = H_3 + H_4 = qR - \frac{8}{21} q f$ (4)

Moments

The Moment Produced By The Uniform Portion of Dead Load is From Eq $\boxed{2}$

$$M_3 = \frac{q f^2}{14}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

The Moment Produced By The Additional Load is As Follows:-

From Eq $\boxed{4A}$ $M_4 = -H_4 y + \frac{q f a^2}{12 R}\left(1 - \frac{x^4}{a^4}\right)$

Eq $\boxed{4B}$ $H_4 = \frac{4}{21} q f$

$$\therefore M_4 = -\frac{4}{21} q f y + \frac{q f a^2}{12 R}\left(1 - \frac{x^4}{a^4}\right)$$

Substitute $f = \frac{a^2}{2R}$ $M_4 = -\frac{4}{21} q f y + \frac{q f^2}{6}\left(1 - \frac{x^4}{a^4}\right)$ By $y = f\left(1 - \frac{x^2}{a^2}\right)$

Substitute $y = f\left(1 - \frac{x^2}{a^2}\right)$ $M_4 = -\frac{8}{42} q f^2\left(1 - \frac{x^2}{a^2}\right) + \frac{7}{42} q f^2\left(1 - \frac{x^4}{a^4}\right)$

$$\therefore M_4 = -\frac{q f^2}{42}\left(8 - \frac{8x^2}{a^2} - 7 + \frac{7x^4}{a^4}\right) = -\frac{q f^2}{42}\left(1 - \frac{8x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

The Total Moment is Therefore $M_D = M_3 + M_4$

$$M_D = \frac{q f^2}{21}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right) \quad \boxed{5}$$

It is Noted That For Moment The Dead Load is Equivalent To $\frac{2}{3}$ of The Same Load Considered As A Live Load. Therefore For Moment Calculation The Dead Load Can Be Considered a Proportional Part of Live Load P (See Eq. $\boxed{2}$ & Eq $\boxed{5}$)

Combined Live & Dead Load

$$H = H_L + H_D = (P+q)\,R - \left(\tfrac{4}{7}P + \tfrac{2}{21}q\right) \tfrac{}{}$$

$$M = M_L + M_D = \frac{1}{14}\left(P + \frac{2}{3}q\right) f^2\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right) \quad [6]$$

To Determine Points of Maximum Positive And Negative Moments Equate The First Derivative of [6] To Zero.

$$\text{or } \frac{d}{dx}\left[\frac{pf^2}{14}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right) + \frac{qf^2}{21}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right)\right] = 0$$

One Value is when $x = 0$ Placing This Value in [6]

$$M = \frac{f^2}{14}\left(P + \frac{2}{3}q\right) \quad [7]$$

Differentiating

$$-\frac{Pf^2}{14} \times \frac{(8)(2)x}{a^2} + \frac{Pf^2(4)x^3}{a^4} - \frac{qf^2}{21}\frac{(8)(2)x}{a^2} + \frac{qf^2}{3}\frac{4x^3}{a^4} = 0$$

Divide Thru By x

$$-\left(\frac{Pf^2}{14}\cdot\frac{16}{a^2}\right) + \left(\frac{Pf^2}{2}\cdot\frac{4x^2}{a^4}\right) - \left(\frac{qf^2}{21}\cdot\frac{16}{a^2}\right) + \left(\frac{qf^2}{3}\cdot\frac{4x^2}{a^4}\right) = 0$$

$$\frac{4x^2}{a^4}\left(\frac{Pf^2}{2} + \frac{qf^2}{3}\right) - \frac{16}{a^2}\left(\frac{Pf^2}{14} + \frac{qf^2}{21}\right) = 0$$

$$\frac{4x^2}{a^4}\left(\frac{21Pf^2}{42} + \frac{14qf^2}{42}\right) - \frac{16}{a^2}\left(\frac{3Pf^2}{42} + \frac{2qf^2}{42}\right) = 0$$

$$4x^2 = \frac{16a^4}{a^2}\;\frac{\frac{3Pf^2}{42} + \frac{2qf^2}{42}}{\frac{21Pf^2}{42} + \frac{14qf^2}{42}} \quad \text{or } X^2 = 4a^2\left(\frac{\frac{1}{42}\left[3Pf^2 + 2qf^2\right]}{\frac{1}{42}\left[21Pf^2 + 14qf^2\right]}\right)$$

$$X^2 = \frac{4a^2}{7} \quad \text{or } X = a\sqrt{\frac{4}{7}}$$

Substitute This Value of X in [6]

$$M = \frac{Pf^2}{14}\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right) + \frac{qf^2}{21}\left(1 - \frac{8x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

$$= \frac{Pf^2}{14}\left[1 - 8\left(\frac{4}{7}\right) + 7\left(\frac{4^2}{7^2}\right)\right] + \frac{qf^2}{21}\left[1 - 8\left(\frac{4}{7}\right) + 7\left(\frac{4^2}{7^2}\right)\right]$$

$$= \frac{9f^2}{7\times 14}\left(P + \frac{2}{3}q\right) = \frac{9}{98}f^2\left(P + \frac{2}{3}q\right) \quad [8]$$

THRUST - DEAD LOAD

The Thrust At Any Arbitrary Point is Equal To

$T_d = \sqrt{H_d^2 + V_{xd}^2}$ $\qquad V_x = qx + \int_0^x q \frac{f}{R} \cdot \frac{X^2}{a^2} dx$

$$\text{or } V_x = qx + q\frac{f}{R} \cdot \frac{X^3}{3a^2}$$

$$T_d = \sqrt{H_d^2 + q^2\left(X + \frac{f}{R}\frac{X^3}{3a^2}\right)^2}$$

THE THRUST AT THE SILLS

Substitute in The Above Equation The Value $qR - \frac{8}{21}qf$ For H_d

Also $a = X$

$$T_o = q\sqrt{\left(R - \frac{8}{21}f\right)^2 + \left(a + \frac{f}{R}\frac{a^3}{3a^2}\right)^2}$$

$$\text{or } T_o = q\sqrt{\left(R - \frac{8}{21}f\right)^2 + a^2\left(1 + \frac{f}{3R}\right)^2} \qquad \text{As Before } a^2 = 2Rf$$

$$= q\sqrt{R^2 - \frac{16}{21}Rf + \frac{64}{441}f^2 + 2Rf\left[1 + \frac{2f}{3R} + \frac{f^2}{9R^2}\right]}$$

$$= q\sqrt{R^2 - \frac{16}{21}Rf + \frac{64}{441}f^2 + 2Rf + \frac{4Rf^2}{3R} + \frac{2Rf^3}{9R^2}}$$

$$= q\sqrt{R^2 + \frac{26}{21}Rf + \frac{652}{441}f^2 + \frac{2f^3}{9R}}$$

$$= q\sqrt{R^2 + \frac{26}{21}Rf + \frac{169}{441}f^2 + \frac{483}{441}f^2 + \frac{2f^3}{9R}}$$

$$= q\sqrt{\left(R + \frac{13}{21}f\right)^2 + \frac{483}{441}f^2 + \frac{2f^3}{9R}}$$

The Last Two Terms Are Neglible In Comparison With The First Term So May Be Neglected (Approx 1%)

$$\therefore T_o = q\left(R + \frac{13}{21}f\right)$$

THRUST - COMBINED

The Total Maximum Thrust At The Sill is With Good Approximation:-

$$T_o\ \text{LIVE} = PR + P\frac{3}{7}f$$

$$T_o\ \text{DEAD} = qR + \frac{13}{21}qf$$

$$T\ \text{Combined} = R(P+q) + f\left(\frac{3}{7}P + \frac{13}{21}q\right)$$

The Minimum Value of The Thrust At The Crown Amounts To:-

$$T\ \text{min.} = H\ \text{Combined} = (P+q)R - \left(\frac{4}{7}P + \frac{8}{21}q\right)f$$

SUMMARY

Of Formulas Obtained From The Preceeding Analysis

$$H = (P+q)R - \left(\frac{4}{7}P + \frac{8}{21}q\right)f$$

$$M_x = \frac{f^2}{14}\left(P + \frac{2}{3}q\right)\left(1 - 8\frac{x^2}{a^2} + 7\frac{x^4}{a^4}\right)$$

$$M_{MAX+} = \frac{f^2}{14}\left(P + \frac{2}{3}q\right)$$

$$M_{MAX-} = -\frac{9}{98}f^2\left(P + \frac{2}{3}q\right)$$

$$T_{COMBINED} = R(P+q) + f\left(\frac{3}{7}P + \frac{13}{21}q\right)$$

$$T_{MINIMUM} = H_{COMBINED} = (P+q)R - \left(\frac{4}{7}P + \frac{8}{21}q\right)f$$

Comparing The Results Obtained By Use of These Formulas And Those Obtained By The Exact Method for a Particular Case Follows:-

Recommended Formulas	Exact Formulas
H = 1584.2 LBS	1584.0 LBS
M_P = 244.0 FT LBS	249.0 FT LBS
M_N = 316.0 FT LBS	317.0 FT LBS

Denotations

L	Span of roof
a	Half span of arches
R	Radius of Arch
D	Distance between ends of building
f	Rise of roof
ℓ	Length of Lamellas
e	Eccentricity of joints
b	Width of Lamellas
h	Height of Lamellas
B	Inclination between Lamellas
s	Width of Diamonds
φ_c	Half central angle of arches
A_s	Cross-section area of sills
A_L	Cross-section area of Lamellas
E	Young's modulus for lumber
p	Live load / sq. foot of Horizontal Projection
q	Dead load / sq. foot of Length of Arc
w	Wind load / sq. foot of Vertical Projection
x	Ordinate from center of chord to point under consideration.
y	Abcissa from the chord to the point under consideration